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Sufficient Stability Criteria and Uniform Stability of Difference Schemes[1]

Boris Yu. Scobelev and Evgenii V. Vorozhtsov

Institute of Theoretical and Applied Mechanics, Russian Academy of Sciences, Novosibirsk 630090, Russia
E-mail: scobelev@online.nsk.su, vorozh@itam.nsc.ru

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We prove two new criteria for the sufficiency of the von Neumann condition for stability of difference schemes. The first criterion is that the von Neumann criterion is sufficient for stability if a finite power of the amplification matrix is a uniformly diagonalizable matrix. The second criterion relaxes the uniform diagonalizability requirement for the amplification matrix: The uniform diagonalizability is needed only in some subregion of the parameter values, and for the remaining parameter values, all the eigenvalues of the amplification matrix should be strictly less than unity in modulus. The numerical investigation of the behavior of the norms of powers of amplification matrix has pointed to the advisability of introducing a new definition, the uniform stability. We prove constructive criteria for uniform stability. We investigate the satisfaction of the obtained uniform stability criteria for a number of well-known difference schemes for the numerical solution of fluid dynamics problems. © 2000 Academic Press

Key Words: finite difference scheme; stability; compressible flow.

1. INTRODUCTION

The Fourier method is the most popular practical method for stability investigation of difference schemes in the quadratic norm. It is well known that the widely accepted von Neumann criterion is only necessary for stability and does not ensure stability of the difference scheme. At present, there is a vast literature devoted to the sufficient stability conditions of difference schemes (see, for example, [1–5]). The most general conditions were obtained in the Kreiss matrix theorem [6]. These conditions, however, proved to be so complex that it was noted by Kreiss himself [7]: “In fact, it is almost hopeless to apply them directly to a practical problem.” Therefore, he has introduced in [7] the notion of the dissipative difference schemes for hyperbolic equations, and he obtained for these schemes stability

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conditions simpler than those of [6]. Other general stability conditions were obtained in the works of Kato [8] and Buchanan [9]. However, they also proved to be too complex for practical application.

The following two particular conditions for the sufficiency of the von Neumann criterion have gained the most widespread acceptance in practice [1]: (1) the amplification matrix G of difference scheme is a normal matrix and (2) the matrix G is a uniformly diagonalizable matrix. A generalization of the first condition was obtained in [10]: It was proved that the von Neumann criterion is sufficient for stability, if a finite power of the amplification matrix G is a normal matrix. In the present work, we consider a possibility for relatively simple extensions of the second condition. We show that the von Neumann criterion is sufficient for stability if a finite power of matrix G is uniformly diagonalizable. In addition, we consider the case where the matrix G is diagonalizable only in some subregion of the values of its arguments, and we formulate the conditions for the sufficiency of the von Neumann criterion. Note that at the formulation of the new criteria for stability of difference schemes, we have paid the closest attention to the feasibility of their practical realization rather than to the universality of these criteria. With the advent of the computer algebra systems, the analytic execution of such procedures as the product of two matrices, the determination of eigenvalues, the check-up of the normality, and the diagonalizability conditions does already not need the execution of bulky hand calculations and is realizable for the amplification matrices of practically important difference schemes.

The numerical computations of $\|G^n\|$ as a function of the number of time steps n have shown that the conventional stability concept is too general in a certain sense, and the difference schemes with qualitatively different properties prove to be stable. For example, if the amplification matrix G is normal, then the quantity $\|G^n\|$ will be a decreasing function of the number of time steps n. On the other hand, the difference scheme for the wave equation from [1] is also stable at the Courant numbers $\kappa < 1$. However, the quantity $\|G^n\|$ is an oscillatory function of n. The amplitude and period of oscillations depend on κ and increase unboundedly as $\kappa \to 1$. It is shown that one can subdivide the stable difference schemes into two groups. The difference schemes for which $\|G^n\|$ is an oscillatory function of n belong to the first group. The second group is formed by the difference schemes, for which $\|G^n\|$ is a nonincreasing function of n. This result shows that it makes sense to introduce an additional characteristic of difference scheme.

In this connection, we propose a new definition for stability, namely, the uniform stability of difference schemes. The conducted investigations of difference schemes have shown that the class of uniformly stable difference schemes is sufficiently wide. Besides the stable schemes with normal amplification matrix, the well-known two-cycle MacCormack scheme and the TVD scheme for the two-dimensional advection equation proved to be uniformly stable. The well-known concepts of strong stability [1] and uniform correctness [3] are the closest ones with respect to the uniform stability concept. It is also interesting to note that one of the conditions for stability of difference schemes with variable coefficients obtained in [11] is indeed one of the uniform stability criteria for schemes with constant coefficients.

The paper is organized as follows. In Section 2, we formulate the difference Cauchy problem as well as the von Neumann criterion and prove two new criteria for the sufficiency of the von Neumann condition.

In Section 3, we carry out a numerical investigation of the behavior of the norms of powers of the amplification matrix for a number of stable difference schemes. In Section 4, we present the definition of uniform stability of difference schemes and prove Theorem 3, which gives a constructive criterion for uniform stability. The investigation of uniform stability of specific difference schemes has shown that it is advisable to introduce the concept of the locally uniform stability, which imposes weaker restrictions on difference scheme. Theorem 4 gives sufficient conditions for the locally uniform stability. In Section 5, we investigate the uniform stability of difference schemes for the Euler equations. In Section 6, we formulate the conclusions.

2. SOME NEW CRITERIA FOR THE SUFFICIENCY OF THE VON NEUMANN CONDITION

Consider the Cauchy problem for the systems of linear differential equations with constant coefficients

$$\frac{\partial \vec{U}}{\partial t} = P\left(\frac{\partial}{\partial x}\right)\vec{U}, \quad t > 0, \quad \vec{U}(x, 0) = \vec{U}_0(x), \tag{1}$$

where $\vec{U} = \{U_1(x,t), \ldots, U_m(x,t)\}$ is the vector function of x and t; $m \geq 1$, $x = (x_1, \ldots, x_L)$, $L \geq 1$ is the number of spatial variables $x_1, \ldots, x_L$, t is the time, $P(\frac{\partial}{\partial x})$ is an $m \times m$ matrix whose elements are the polynomials in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_L}$, and $\vec{U}_0(x)$ is a given vector function.

Let us approximate the system (1) by a $(q+1)$th-level difference scheme, $q \geq 1$. If $q > 1$, then the system of $(q+1)$th-level difference equations may be reduced to a system of two-level equations

$$C_1\vec{u}^{\,n+1} + C_2\vec{u}^{\,n} = 0 \tag{2}$$

by introducing the new dependent variables [1], where $\vec{u}^{\,n}$ is the difference solution vector containing $m + q - 1$ components, C_1 and C_2 are some linear (generally matrix) operators with constant coefficients, which depend on the time step τ and on the steps $h_1, \ldots, h_L$ of the uniform computing mesh along the axes $x_1, \ldots, x_L$, respectively; $\vec{u}^{\,n} = \vec{u}(\vec{x}, n\tau)$, and n is the number of time level, $n = 0, 1, \ldots, [T/\tau]$; the symbol $[a]$ denotes the integral part of the number a.

Assume that the operator C_1 in (2) is invertible. Then we can solve the system of equations (2) with respect to $\vec{u}^{\,n+1}$,

$$\vec{u}^{\,n+1} = S\vec{u}^{\,n}, \quad n = 0, 1, \ldots, [T/\tau], \tag{3}$$

where S is the step operator of difference scheme, $S = -C_1^{-1}C_2$. The initial condition for difference scheme (3) is determined from (1)

$$\vec{u}^{\,0} = \vec{u}_0(\vec{x}), \tag{4}$$

where $\vec{u}_0(\vec{x})$ is a given vector function.

As is known, the von Neumann criterion [1–4] is the necessary condition for stability of the difference Cauchy problem (3), (4),

$$\max_{\vec{\xi},i}|\lambda_i(\vec{\kappa}, \vec{\xi})| \leq 1 + O(\tau), \tag{5}$$

where $\vec{\xi} = \vec{k} \cdot \vec{h}$ is the wave vector; $\vec{k} = (k_1, \ldots, k_L)$ are the real wavenumbers; $\vec{h} = (h_1, \ldots, h_L)$, $\vec{\kappa} = (\kappa_1, \ldots, \kappa_M)$, where $\kappa_m,\ m = 1, \ldots, M\ (M \geq 1)$ are the nondimensional similarity parameters in the space of variation of which the stability region of difference scheme is determined; and $\lambda_i(\vec{\kappa}, \vec{\xi})$ are the eigenvalues of the amplification matrix G, which is the Fourier symbol of the step operator S entering difference scheme (3).

It is well known [1, 4] that the von Neumann condition is not only necessary but also sufficient for stability of scheme if the amplification matrix G is a normal matrix (or, which is the same, the step operator S is a normal operator):

$$G^*G = GG^* \quad (S^*S = SS^*).$$

An extension of this sufficient condition for stability was obtained in [10] for the case where there is an integer $N \geq 1$ such that G^N is a normal matrix.

It is well known that the class of normal matrices is relatively narrow. The class of diagonalizable matrices is a wider class, because the class of diagonalizable matrices includes the class of normal matrices. That is, each normal matrix is also a diagonalizable matrix. The next theorem can therefore be considered a generalization of the theorem from [10]. In addition, the theorem below generalizes another well-known criterion for the sufficiency of the von Neumann criterion [1]: the case, where the matrix G is a uniformly diagonalizable matrix.

THEOREM 1. *Let the following conditions be satisfied.*

1. The step operator S is uniformly bounded in some region of the parameters $\vec{\kappa} \in D$:

$$\|S\| \leq M_1, \quad \vec{\kappa} \in D.$$

2. For the Fourier symbol $G(\vec{\kappa}, \vec{\xi})$ of the step operator S, there exists such a positive integer $N \geq 1$ that $G^N(\vec{\kappa}, \vec{\xi})$ is a uniformly diagonalizable matrix at all $\vec{\xi}$ and the parameter values $\vec{\kappa} \in D$. Then the von Neumann condition for the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ is the necessary and sufficient condition for the difference scheme stability for $\vec{\kappa} \in D$.

Proof. Consider the definition for the difference scheme stability [1–3]. The difference scheme is called stable if there exists such a number $M(\overline{t}) > 0$ that

$$\|C_{n,k}\| \leq M(\overline{t}) \tag{6}$$

for all $0 \leq k \leq n - 1$ and $n\tau \leq \overline{t}$, where $C_{n,k}$ is the transition operator. For the difference scheme with constant coefficients $C_{n,k} = S^{n-k}$; therefore, the difference scheme will be stable if the inequality

$$\|S^{n-k}\| \leq M(\overline{t}) \tag{7}$$

is satisfied for all n and $0 \leq k \leq n - 1$.

As follows from Condition 2, there exists a similarity transformation $R(\vec{\kappa}, \vec{\xi})$ such that

$$G^N(\vec{\kappa}, \vec{\xi}) = R(\vec{\kappa}, \vec{\xi})T(\vec{\kappa}, \vec{\xi})R^{-1}(\vec{\kappa}, \vec{\xi}), \tag{8}$$

where $T(\vec{\kappa}, \vec{\xi})$ is a diagonal matrix whose diagonal is the eigenvalues $\zeta_i(\vec{\kappa}, \vec{\xi})$ of matrix $G^N(\vec{\kappa}, \vec{\xi})$, and

$$\zeta_i(\vec{\kappa}, \vec{\xi}) = \lambda_i^N(\vec{\kappa}, \vec{\xi}), \tag{9}$$

where $\lambda_i(\vec{\kappa}, \vec{\xi})$ are the eigenvalues of matrix $G(\vec{\kappa}, \vec{\xi})$. It is well known [12] that the columns of the transformation matrix $R(\vec{\kappa}, \vec{\xi})$ are the generalized eigenvectors of the amplification matrix $G^N(\vec{\kappa}, \vec{\xi})$. Therefore, one can always choose $R(\vec{\kappa}, \vec{\xi})$ in such a way that the following estimates will be valid:

$$\max_{\vec{\xi}} \|R(\vec{\kappa}, \vec{\xi})\| \le C \quad \text{and} \quad \max_{\vec{\xi}} \|R^{-1}(\vec{\kappa}, \vec{\xi})\| \le C, \quad \vec{\kappa} \in D. \tag{10}$$

Let us estimate the norm of operator S^{n-k} $(0 \le k \le n-1)$:

$$\|S^{n-k}\| = \|S^{pN+\delta}\| \le \|S^{pN}\| \|S^{\delta}\|, \tag{11}$$

where $p = [\frac{n-k}{N}]$ and $\delta = (n-k) - N[\frac{n-k}{N}]$. With regard for condition 1 of the theorem we obtain

$$\|S(\vec{\kappa})^{\delta}\| \le \max\{1, \|S(\vec{\kappa})^{N-1}\|\} \le M_2 = \max\left\{1, M_1^{N-1}\right\}, \quad \vec{\kappa} \in D. \tag{12}$$

Let us estimate the first factor on the right-hand side of inequality (11):

$$\begin{aligned} \|S(\vec{\kappa})^{pN}\| &= \max_{\vec{\xi}} \|G^{pN}(\vec{\kappa}, \vec{\xi})\| = \max_{\vec{\xi}} \|R(\vec{\kappa}, \vec{\xi}) T^p(\vec{\kappa}, \vec{\xi}) R^{-1}(\vec{\kappa}, \vec{\xi})\| \\ &\le C^2 \max_{\vec{\xi}} \|T^p(\vec{\kappa}, \vec{\xi})\| \le C^2 \left[\max_{\vec{\xi}} \|T(\vec{\kappa}, \vec{\xi})\|\right]^p, \quad \vec{\kappa} \in D. \end{aligned} \tag{13}$$

We have used the estimates (10) at the derivation of this inequality.

If the von Neumann condition is satisfied, then $\max_{\vec{\xi}} |\lambda_i(\vec{\kappa}, \vec{\xi})| \le 1$ and consequently

$$\max_{\vec{\xi}, i} |\zeta_i(\vec{\kappa}, \vec{\xi})| = \max_{\vec{\xi}, i} |\lambda_i(\vec{\kappa}, \vec{\xi})|^N \le 1.$$

Because $T(\vec{\kappa}, \vec{\xi})$ is a diagonal matrix we have

$$\max_{\vec{\xi}} \|T(\vec{\kappa}, \vec{\xi})\| = \max_{\vec{\xi}, i} |\zeta_i(\vec{\kappa}, \vec{\xi})| \le 1. \tag{14}$$

Substituting the estimates (12)–(14) in inequality (11), we obtain

$$\|S^{n-k}\| \le M_2 C^2 \tag{15}$$

for any n, $0 \le k \le n-1$ and $\vec{\kappa} \in D$.

This completes the proof of the theorem. ■

Now consider the difference schemes for which it is impossible to find such an integer $N \ge 1$ that G^N is a normal or uniformly diagonalizable matrix. Denote by Ω the region of periodicity in $\vec{\xi}$ of the amplification matrix $G(\vec{\kappa}, \vec{\xi})$:

$$\Omega = [0, 2\pi]^L = \overbrace{[0, 2\pi] \times [0, 2\pi] \times \cdots \times [0, 2\pi]}^{L}. \tag{16}$$

THEOREM 2. *Assume that the norm of the step operator $S(\vec{\kappa})$ is uniformly bounded,*

$$\|S\| \leq M_1, \quad \vec{\kappa} \in D, \tag{17}$$

and the von Neumann condition is satisfied in some region of the parameters $\vec{\kappa} \in D$.

For a given $\vec{\kappa}$ denote by $\Omega_\delta(\vec{\kappa})$ the set of the values of vector $\xi \in \Omega$, such that

$$\max_i |\lambda_i(\vec{\kappa}, \vec{\xi})| < 1 - \delta, \quad \vec{\xi} \in \Omega_\delta(\vec{\kappa}), \tag{18}$$

where $\lambda_i(\vec{\kappa}, \vec{\xi})$ are the eigenvalues of the amplification matrix $G(\vec{\kappa}, \vec{\xi})$, and $\delta > 0$ is a constant.

Assume that there is such a $\delta_0 > 0$ that the matrix $G(\vec{\kappa}, \vec{\xi})$ is uniformly diagonalizable for $\xi \in \Omega \backslash \Omega_{\delta_0}(\vec{\kappa})$ and all $\vec{\kappa} \in D$ ($\Omega \backslash \Omega_{\delta_0}(\vec{\kappa})$ is a complement of the set $\Omega_{\delta_0}(\vec{\kappa})$ in the set Ω).

Then the difference scheme is stable in D.

Proof. In order to prove the theorem we must show that the quantity

$$\|S^p(\vec{\kappa})\| = \max_{\vec{\xi}} \|G^p(\vec{\kappa}, \vec{\xi})\| \tag{19}$$

is uniformly bounded for any integer $p \geq 1$ and $\vec{\kappa} \in D$. In accordance with the conditions of the theorem, we rewrite the right-hand side of (19) as

$$\max_{\vec{\xi}} \|G^p(\vec{\kappa}, \vec{\xi})\| = \max\left\{ \max_{\vec{\xi} \in \Omega \backslash \Omega_{\delta_0}} \|G^p(\vec{\kappa}, \vec{\xi})\|, \max_{\vec{\xi} \in \Omega_{\delta_0}} \|G^p(\vec{\kappa}, \vec{\xi})\| \right\}. \tag{20}$$

1. Consider the first item within the braces in (20). Because the von Neumann criterion is satisfied in accordance with the conditions of the theorem, and the matrix $G(\vec{\kappa}, \vec{\xi})$ is uniformly diagonalizable at $\vec{\xi} \in \Omega \backslash \Omega_{\delta_0}(\vec{\kappa})$,

$$\max_{\vec{\xi} \in \Omega \backslash \Omega_{\delta_0}} \|G^p(\vec{\kappa}, \vec{\xi})\| \leq M_0 \tag{21}$$

for any $p \geq 1$ and $\vec{\kappa} \in D$ (M_0 is a constant that does not depend on $\vec{\kappa}$). It is easy to prove inequality (21) by using the proof of Theorem 1 at $N = 1$ (see also the proof of the sufficiency of the von Neumann criterion for difference schemes with a uniformly diagonalizable amplification matrix in [1]).

2. We now prove the uniform boundedness of the second item within the braces in formula (20):

$$\max_{\vec{\xi} \in \Omega_{\delta_0}} \|G^p(\vec{\kappa}, \vec{\xi})\|. \tag{22}$$

Consider the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ and make use of the theorem on the spectral decomposition of operator [12] and the theorem on the Jordan form of a matrix. It follows from these theorems that there exists a similarity transformation $V(\vec{\kappa}, \vec{\xi})$, which reduces the matrix $G(\vec{\kappa}, \vec{\xi})$ to the Jordan form,

$$V(\vec{\kappa}, \vec{\xi}) G(\vec{\kappa}, \vec{\xi}) V(\vec{\kappa}, \vec{\xi})^{-1} = T(\vec{\kappa}, \vec{\xi}) + D(\vec{\kappa}, \vec{\xi}), \tag{23}$$

where $T(\vec{\kappa}, \vec{\xi})$ is a diagonal matrix, and $D(\vec{\kappa}, \vec{\xi})$ is a nilpotent matrix. The entries on the diagonal of matrix $T(\vec{\kappa}, \vec{\xi})$ are the eigenvalues of matrix G (each eigenvalue λ_i is repeated m_i times, where m_i is the multiplicity of λ_i). We also point to the fact that the matrices $T(\vec{\kappa}, \vec{\xi})$ and $D(\vec{\kappa}, \vec{\xi})$ commute.

As was pointed out in the proof of Theorem 1, one can always choose $V(\vec{\kappa}, \vec{\xi})$ in such a way that the estimates

$$\|V(\vec{\kappa}, \vec{\xi})\| \leq C_1 \quad \text{and} \quad \|V^{-1}(\vec{\kappa}, \vec{\xi})\| \leq C_1 \tag{24}$$

will be valid uniformly in $\vec{\xi}$. Because the diagonal matrix $T(\vec{\kappa}, \vec{\xi})$ is a normal operator,

$$\|T(\vec{\kappa}, \vec{\xi})\| = \max_i |\lambda_i(\vec{\kappa}, \vec{\xi})|.$$

Therefore, the estimate

$$\max_{\vec{\xi}} \|T(\vec{\kappa}, \vec{\xi})\| < 1 - \delta_0 \tag{25}$$

follows from the conditions of the theorem. Consider the quantity

$$\max_{\vec{\xi} \in \Omega_{\delta_0}} \|D(\vec{\kappa}, \vec{\xi})\|.$$

Taking the formulas (23)–(25) as well as the boundedness of the step operator S into account, we obtain

$$\begin{aligned} \max_{\vec{\xi} \in \Omega_{\delta_0}} \|D(\vec{\kappa}, \vec{\xi})\| &\leq \max_{\vec{\xi}} [\|T(\vec{\kappa}, \vec{\xi})\| + \|V(\vec{\kappa}, \vec{\xi})\| \|G(\vec{\kappa}, \vec{\xi})\| \|V^{-1}(\vec{\kappa}, \vec{\xi})\|] \\ &\leq 1 + C_1^2 M_1 \equiv M_2. \end{aligned} \tag{26}$$

Write (22) as

$$\begin{aligned} \max_{\vec{\xi} \in \Omega_{\delta_0}} \|G^p(\vec{\kappa}, \vec{\xi})\| &= \max_{\vec{\xi} \in \Omega_{\delta_0}} \|V^{-1}(\vec{\kappa}, \vec{\xi}) V(\vec{\kappa}, \vec{\xi}) G^p(\vec{\kappa}, \vec{\xi}) V^{-1}(\vec{\kappa}, \vec{\xi}) V(\vec{\kappa}, \vec{\xi})\| \\ &\leq C_1^2 \max_{\vec{\xi}} \|V(\vec{\kappa}, \vec{\xi}) G^p(\vec{\kappa}, \vec{\xi}) V^{-1}(\vec{\kappa}, \vec{\xi})\| \\ &= C_1^2 \max_{\vec{\xi}} \|(T(\vec{\kappa}, \vec{\xi}) + D(\vec{\kappa}, \vec{\xi}))^p\|. \end{aligned} \tag{27}$$

For making the further estimations we take into account the fact that D is a nilpotent operator. That is, $D^l = 0$ at some natural $1 < l \leq m$, where m is the dimension of the amplification matrix G. In addition, we make use of the commutativity of operators T and D. Then, at sufficiently large values of p ($p \geq m$) we obtain

$$\begin{aligned} \|(T + D)^p\| &= \left\| \sum_{j=0}^{m-1} \binom{p}{j} T^{p-j} D^j \right\| \\ &= p^{m-1} \left\| T^{p-m+1} \left[\sum_{j=0}^{m-1} \frac{1}{p^{m-1}} \binom{p}{j} T^{m-1-j} D^j \right] \right\| \\ &\leq p^{m-1} \|T\|^{p-m+1} \left[\sum_{j=0}^{m-1} \frac{1}{p^{m-1}} \binom{p}{j} \|T\|^{m-1-j} \|D\|^j \right], \end{aligned} \tag{28}$$

where $\binom{p}{j}$ are the binomial coefficients. Introduce the notation

$$\max_{\vec{\xi}\in\Omega_{\delta_0}} \|T(\vec{\kappa},\vec{\xi})\| = \overline{\lambda} \tag{29}$$

and take the inequalities (25) and (26) into account. Then the substitution of inequality (28) in (27) yields the inequality

$$\max_{\vec{\xi}\in\Omega_{\delta_0}} \|G^p(\vec{\kappa},\vec{\xi})\| \le C_1^2 p^{m-1}\overline{\lambda}^{p-m+1}\left[\sum_{j=0}^{m-1} \frac{1}{p^{m-1}}\binom{p}{j} M_2^j\right]. \tag{30}$$

It is easy to see that the expression in square brackets is uniformly bounded at any $p \ge m$. Therefore,

$$\max_{\vec{\xi}\in\Omega_{\delta_0}} \|G^p(\vec{\kappa},\vec{\xi})\| \le M_3 p^{m-1}\overline{\lambda}^{p-m+1}. \tag{31}$$

Because $\overline{\lambda} < 1-\delta_0$, the right-hand side of inequality (31) tends to zero as $p \to \infty$. Therefore, the quantity on the left-hand side of inequality (31) is uniformly bounded at all $p \ge m$. The boundedness of this quantity at $p < m$ follows from the uniform boundedness of the norm of the step operator S (17). Consequently,

$$\max_{\vec{\xi}\in\Omega_{\delta_0}} \|G^p(\vec{\kappa},\vec{\xi})\| \le M_4. \tag{32}$$

Substituting the estimates (21) and (32) into formula (20), we obtain that the quantity $\max_{\xi} \|G^p(\vec{\kappa},\vec{\xi})\|$ is uniformly bounded at any integer $p \ge 1$ and $\vec{\kappa} \in D$. The theorem is proved. ∎

EXAMPLE 1. Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{33}$$

where $\nu = \text{const} > 0$. Let us approximate (33) by the following explicit three-level scheme [13]:

$$\frac{u_j^{n-1} - 4u_j^n + 3u_j^{n+1}}{2\tau} = \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \quad n = 1, 2, \ldots. \tag{34}$$

Introducing the auxiliary dependent variable $v^n = u^{n+1}$ we can rewrite the difference scheme (34) as a system of two two-level difference equations:

$$\begin{gathered} u_j^{n+1} = v_j^n; \\ 3v_j^{n+1} - 4v_j^n + u_j^n = 2\frac{\nu\tau}{h^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right). \end{gathered} \tag{35}$$

The amplification matrix G obtained by the Fourier method from system (35) has the form

$$G = \begin{pmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{4}{3} - \frac{8}{3}a \end{pmatrix}, \tag{36}$$

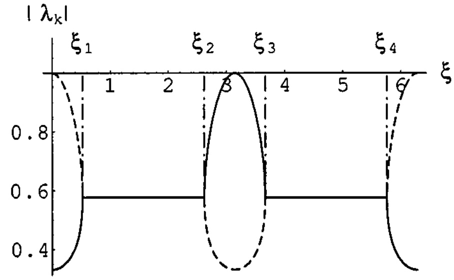

FIG. 1. Fletcher's scheme (34). The absolute values of eigenvalues as functions of ξ for $\kappa = 1$: (– – –) $|\lambda_1(\xi)|$, (——) $|\lambda_2(\xi)|$.

where

$$a = \kappa \sin^2 \frac{\xi}{2}, \quad \kappa = \frac{\nu\tau}{h^2}. \tag{37}$$

It follows from (36) that the characteristic equation of scheme (34) is a quadratic equation with real coefficients. The roots λ_1, λ_2 of this equation are given by formulas

$$\lambda_1 = \frac{1}{3}(2 - 4a + \sqrt{1 - 16a + 16a^2}), \quad \lambda_2 = \frac{1}{3}(2 - 4a - \sqrt{1 - 16a + 16a^2}). \tag{38}$$

An analysis of the roots (38) shows that the von Neumann condition is satisfied for $\kappa \le 1$. For the values $\kappa < \frac{1}{4}(2 - \sqrt{3})$, the zeroes λ_1, λ_2 are real and different for all ξ; hence, the matrix G is diagonalizable and the difference scheme (34) is stable.

For $\frac{1}{4}(2 - \sqrt{3}) \le \kappa \le 1$ there exist, in the general case, four values of the parameter ξ at which the eigenvalues (38) degenerate, and the Jordan form of matrix G has a nonzero nilpotent part (see Fig. 1). Let us show that the conditions of Theorem 2 are satisfied in this region of the values of the parameter κ.

Note that the norm of all degenerate eigenvalues $\lambda_\alpha(\xi(\kappa))$ is equal to $1/\sqrt{3}$. Therefore, one can take as δ_0 in Theorem 2 any quantity from the interval $0 < \delta_0 < 1 - \frac{1}{\sqrt{3}} - \varepsilon$ ($\varepsilon > 0$ is arbitrarily small). We take $\delta_0 = 0.1$ for definiteness. It is easy to show that at $(1/4)(2 - \sqrt{3}) \le \kappa < \kappa_0$ ($\kappa_0 \approx 0.976$) the set $\Omega\backslash\Omega_{\delta_0}(\kappa)$ consists of two parts, $\Omega\backslash\Omega_{\delta_0}(\kappa) = [0, \xi_1(\kappa)] \cup [2\pi - \xi_1(\kappa), 2\pi]$, where $\xi_1 \approx 2 \arcsin \frac{0.15}{\sqrt{\kappa}}$. At $\kappa_0 \le \kappa \le 1$ the third part appears,

$$\Omega\backslash\Omega_{\delta_0}(\kappa) = [0, \xi_1(\kappa)] \cup [2\pi - \xi_1(\kappa), 2\pi] \cup [\xi_2(\kappa), 2\pi - \xi_2(\kappa)],$$

where $\xi_2 = 2 \arcsin \sqrt{\frac{\kappa_0}{\kappa}}$. At all values $(1/4)(2 - \sqrt{3}) \le \kappa \le 1$, the amplification matrix G has two different eigenvalues in the region $\Omega\backslash\Omega_{\delta_0}(\kappa)$ and hence is uniformly diagonalizable. The conditions of Theorem 2 are satisfied.

Thus, the von Neumann criterion is a sufficient criterion for stability of scheme (34).

3. THE BEHAVIOR OF THE NORMS OF POWERS OF THE AMPLIFICATION MATRICES FOR STABLE DIFFERENCE SCHEMES

It follows from the stability definition of difference scheme that $\|G^n\|$ must be uniformly bounded for any integer $n \ge 1$. However, the character of stability, that is, the behavior of the transition operator $C_{n,k}$ as a function of the number of time steps n, may be qualitatively different for stable difference schemes. Let us illustrate this at a number of examples.

EXAMPLE 2. Let us approximate the wave equation $u_{tt} = c^2 u_{xx}$ ($c^2 = \text{const} > 0$) by the following scheme [1]:

$$\frac{v_j^{n+1} - v_j^n}{\tau} = c\frac{w_{j+1/2}^n - w_{j-1/2}^n}{h}; \quad \frac{w_{j-1/2}^{n+1} - w_{j-1/2}^n}{\tau} = c\frac{v_j^{n+1} - v_{j-1}^{n+1}}{h}. \tag{39}$$

Here, $w = cu_x,\ v = u_t$. Introducing the vector $\vec{U} = (v_j^n, w_j^n)^T$, we obtain the amplification matrix G of the form [1]

$$G = \begin{pmatrix} 1 & ia \\ ia & 1 - a^2 \end{pmatrix}, \tag{40}$$

where $a = 2\kappa \sin(\xi_1/2)$, and $\kappa = c\tau/h$. As is known [1], the difference scheme (39) is stable at the values of the Courant number $\kappa < 1$ and is weakly stable at $\kappa = 1$ [14].

Consider the behavior of the quantity $\|G^n\|$ at $\kappa < 1$ (that is, in the stability region). Assume $\xi_1 = \pi$ for definiteness. To compute the $\|G^n(\kappa, \pi)\|$ let us make use of the formula

$$\|G^n(\kappa, \pi)\| = \|(G^n)^* G^n\|^{1/2}. \tag{41}$$

As can be seen from Figs. 2a–2d, there is an oscillatory variation in the quantity $\|G^n(\kappa, \pi)\|$, and both the amplitude of oscillations and the mean value of $\|G^n(\kappa, \pi)\|$ increase as $\kappa \to 1$ (the period of oscillations also increases). At $\kappa = 1$ (Fig. 2e), there is a power law growth of the quantity $\|G^n(\kappa, \pi)\|$ as n increases in accordance with the definition of weak stability ($\|G^n\| \sim n^\alpha, \alpha = 1$).

EXAMPLE 3. Consider the system of two-dimensional acoustics equations [1]:

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x_1} = 0, \quad \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x_2} = 0, \quad \frac{\partial p}{\partial t} + \rho_0 c_0^2 \left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} \right) = 0. \tag{42}$$

Here, ρ_0 is the gas density, p is the pressure, c_0 is the sound velocity, and u, v are the components of the gas velocity vector along the x_1, x_2 axes, respectively. Rewrite system (42) in vector matrix form,

$$\frac{\partial \vec{U}}{\partial t} + A\frac{\partial \vec{U}}{\partial x_1} + B\frac{\partial \vec{U}}{\partial x_2} = 0, \tag{43}$$

where

$$\vec{U} = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & 0 \\ \rho_0 c_0^2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho_0} \\ 0 & \rho_0 c_0^2 & 0 \end{pmatrix}. \tag{44}$$

Let us approximate system (43) with the aid of the following three-stage scheme of the Runge–Kutta type [15]:

$$\begin{aligned} &\vec{u}^{(0)} = \vec{u}^{n}, \quad \vec{u}^{(1)} = \vec{u}^{(0)} - \alpha_1 \tau P_h \vec{u}^{(0)}, \\ &\vec{u}^{(2)} = \vec{u}^{(0)} - \alpha_2 \tau P_h \vec{u}^{(1)}, \\ &\vec{u}^{(3)} = \vec{u}^{(0)} - \tau P_h \vec{u}^{(2)}, \quad \vec{u}^{n+1} = \vec{u}^{(3)}. \end{aligned} \tag{45}$$

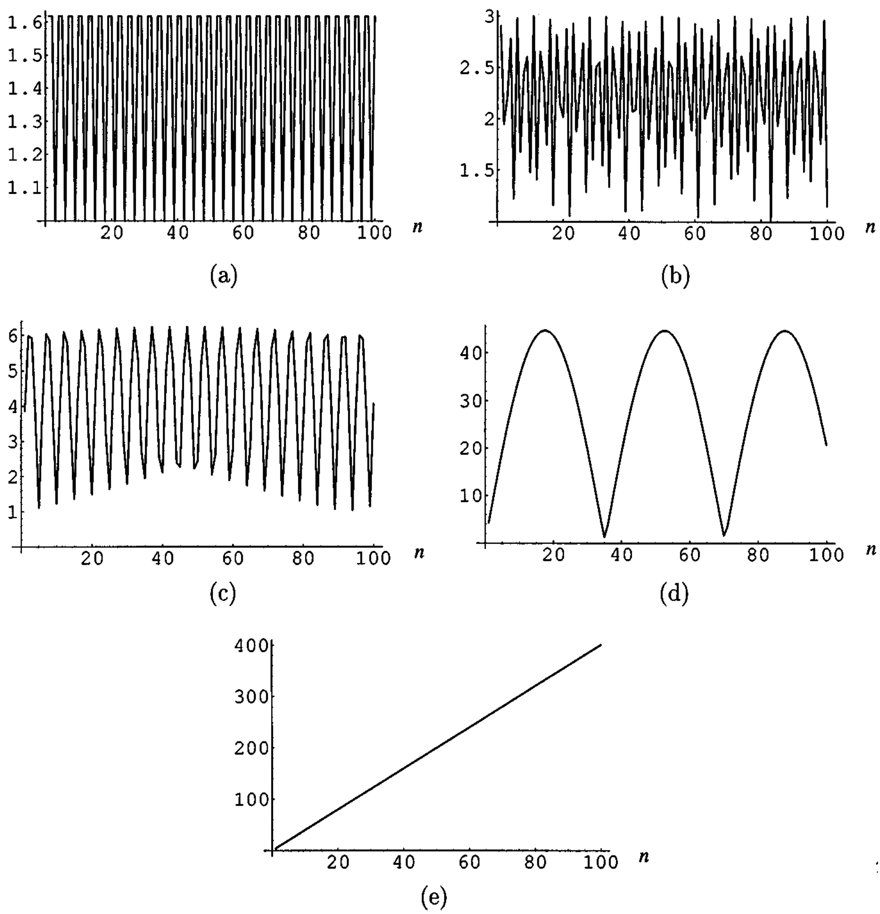

FIG. 2. The graphs of the quantity $\|G^n(\kappa, \pi)\|$ as functions of n for fixed values of κ: (a) $\kappa = 0.5$; (b) $\kappa = 0.8$; (c) $\kappa = 0.95$; (d) $\kappa = 0.999$; (e) $\kappa = 1.0$.

Here, $P_h\vec{u}$ is the difference operator approximating the operator of spatial differentiation in system (43) of the form

$$P\vec{U} = A\frac{\partial \vec{U}}{\partial x_1} + B\frac{\partial \vec{U}}{\partial x_2}.$$

We take for definiteness the approximation by central differences:

$$P_h\vec{u}_{jk} = A\frac{\vec{u}_{j+1,k} - \vec{u}_{j-1,k}}{2h_1} + B\frac{\vec{u}_{j,k+1} - \vec{u}_{j,k-1}}{2h_2}. \tag{46}$$

The quantities α_1, α_2 in (45) are the weight parameters. The values $\alpha_1 = \alpha_2 = \frac{1}{2}$ ensure the second approximation order of scheme (45), in time [15]. Eliminating the intermediate quantities $\vec{u}^{\,(1)}$ and $\vec{u}^{\,(2)}$ from (45), we obtain a two-level difference scheme,

$$\vec{u}^{\,n+1} = S\vec{u}^{\,n},$$

where the step operator S has the form

$$S = I - \tau P_h + \alpha_2 (\tau P_h)^2 - \alpha_1 \alpha_2 (\tau P_h)^3. \tag{47}$$

The amplification matrix corresponding to operator (47) is

$$G = I - iZ - \alpha_2 Z^2 + i\alpha_1 \alpha_2 Z^3, \tag{48}$$

where Z is the result of the Fourier transformation of operator τP_h,

$$Z = d_1 A + d_2 B, \tag{49}$$

$d_\alpha = (\tau / h_\alpha) \sin \xi_\alpha$, and $\alpha = 1, 2$. The eigenvalues of matrix (49) are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm c_0 \sqrt{d_1^2 + d_2^2}.$$

Let μ_l, $l = 1, 2, 3$, be the eigenvalues of matrix G. Then

$$\mu_1 = 1, \quad \mu_{2,3} = 1 - \alpha_2 c_0^2 \left(d_1^2 + d_2^2\right) \mp i c_0 \sqrt{d_1^2 + d_2^2}\, \left[1 - \alpha_1 \alpha_2 c_0^2 \left(d_1^2 + d_2^2\right)\right]. \tag{50}$$

From the inequalities $|\mu_l| \le 1,\ l = 1, 2, 3$, we obtain the von Neumann stability condition in the form

$$\sqrt{\kappa_1^2 + \kappa_2^2} \le \frac{1}{\alpha_1} \left[\frac{2\alpha_1 - \alpha_2 + \left(\alpha_2 \left(\alpha_2 - 4\alpha_1 + 8\alpha_1^2\right)\right)^{0.5}}{2\alpha_2} \right]^{0.5}, \tag{51}$$

where $\kappa_1 = c_0 \tau / h_1$, and $\kappa_2 = c_0 \tau / h_2$. At $\alpha_1 = \alpha_2 = 0.5$ we obtain from this the condition

$$\sqrt{\kappa_1^2 + \kappa_2^2} \le 2. \tag{52}$$

Let us prove that the condition (51) is the sufficient stability condition for scheme (45), (46). It is easy to see that the eigenvalues of matrix G are different at almost all values of ξ_1, ξ_2, except for the points $\xi_1 = \xi_2 = k\pi$ $(k = 0, 1, \ldots)$. At points $\xi_1 = \xi_2 = k\pi$, the matrix G coincides with the identity matrix. Therefore, matrix G is unformly diagonalizable. Consequently, the von Neumann condition (51) is the sufficient stability condition for difference scheme (45), (46).

The numerical computations of the quantity $\|G^n\|$ for $\alpha_1 = \alpha_2 = 0.5$ in (45) and different values of κ_1, κ_2 and ξ_1, ξ_2 are presented in Fig. 3. It can be seen that as in Example 2, an oscillatory behavior of the quantity $\|G^n\|$ takes place. The characteristics of oscillations depend both on the values of the Courant numbers κ_1, κ_2 and on the values ξ_1, ξ_2.

We have presented above the examples of difference schemes, for which the behavior of the quantity $\|G^n\|$ as a function of n is oscillatory. There is, however, a class of difference schemes for which the quantity $\|G^n\|$ is a nonincreasing function of n.

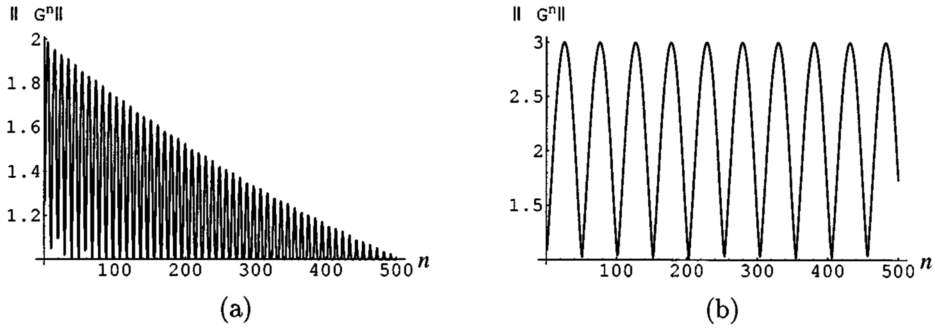

FIG. 3. The Runge–Kutta scheme (45). The graphs of $\|G^n\|$ as functions of the number n of time steps: (a) $\kappa_1 = 0.8,\ \kappa_2 = 1.0, \xi_1 = 0.96\pi,\ \xi_2 = 0.1\pi, \rho_0 c_0 = 2.0$; (b) $\kappa_1 = 1.7,\ \kappa_2 = 1.0, \xi_1 = \xi_2 = 0.99\pi, \rho_0 c_0 = \frac{1}{3}$.

EXAMPLE 4. We again consider the difference scheme from Example 3. It is well known (see, for example, [16]) that the system (42) may be symmetrized by introducing a new vector of dependent variables $\vec{v}$ by formula $\vec{v} = L\vec{U}$, where

$$L = \begin{pmatrix} \sqrt{\rho_0} & 0 & 0 \\ 0 & \sqrt{\rho_0} & 0 \\ 0 & 0 & \frac{1}{c_0\sqrt{\rho_0}} \end{pmatrix}. \tag{53}$$

From (43) we obtain for vector $\vec{v}$ the system

$$\frac{\partial \vec{v}}{\partial t} + A_1 \frac{\partial \vec{v}}{\partial x_1} + B_1 \frac{\partial \vec{v}}{\partial x_2} = 0, \tag{54}$$

where

$$A_1 = LAL^{-1} = \begin{pmatrix} 0 & 0 & c_0 \\ 0 & 0 & 0 \\ c_0 & 0 & 0 \end{pmatrix}, \quad B_1 = LBL^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_0 \\ 0 & c_0 & 0 \end{pmatrix}; \tag{55}$$

thus, matrices A_1 and B_1 are symmetric. Let us replace $\vec{U}$ with $\vec{v}$ and approximate system (54) with the aid of the three-stage Runge–Kutta scheme (45). Because the matrices A_1 and B_1 are obtained from A and B with the aid of the similarity transformations (55), the form of the von Neumann stability condition (51) does not change in new variables $\vec{v}$. The matrix Z in (48) now has the form

$$Z = d_1 A_1 + d_2 B_1 = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ a_1 & a_2 & 0 \end{pmatrix},$$

where $a_k = \kappa_k \sin \xi_k,\ k = 1, 2$. Thus, matrix Z is a real symmetric matrix; hence, the matrix G (48) is also normal [17]. Therefore, the von Neumann necessary stability condition (51) is also sufficient for stability of scheme (45) approximating the symmetric system (54). We present in Fig. 4 the graph of $\|G^n\|$ as a function of n. It can be seen that $\|G^n\| = 1$ for all n. The same behavior of $\|G^n\|$ takes place at other values of κ_1, κ_2, and ξ_1, ξ_2.

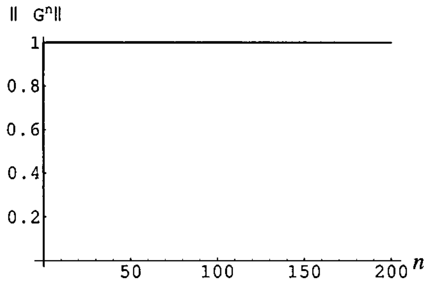

FIG. 4. The Runge–Kutta scheme (45) for the symmetrized acoustics equations (54). The graph of $\|G^n\|$ as a function of the number n of time steps for $\kappa_1 = 0.8,\ \kappa_2 = 1.0,\ \xi_1 = 0.96\pi,\ \xi_2 = 0.1\pi$.

It is interesting to note that the maximum amplitudes of the oscillations of $\|G^n\|$ for the nonsymmetrized difference scheme (45), (46) may easily be estimated by using the following considerations. With regard for (55) we can write the expression for matrix (48) as $G = L^{-1}G_0L$, where G_0 is a normal matrix, and $\|G_0\| \le 1$ under the satisfaction of the von Neumann condition (51). Then the following estimates are valid with regard for (53):

$$\begin{aligned} \|G^n\| &= \left\|L^{-1}G_0^nL\right\| \le \|L^{-1}\| \cdot \|L\| \\ &= \left[\frac{1}{\sqrt{\rho_0}}\max(1, \rho_0c_0)\right] \cdot \left[\sqrt{\rho_0}\max\left(1, \frac{1}{\rho_0c_0}\right)\right] \\ &= \max(1, \rho_0c_0) \cdot \max\left(1, \frac{1}{\rho_0c_0}\right). \end{aligned} \tag{56}$$

Note that the computational results presented in Fig. 3 agree with estimate (56): the magnitude of the quantity $\|G^n\|$ does not exceed the right-hand side of (56).

EXAMPLE 5. Consider the Harten scheme [18] (a TVD variant of the Lax–Wendroff scheme from [1]) for the two-dimensional advection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x_1} + B\frac{\partial u}{\partial x_2} = 0, \tag{57}$$

where A and B are the scalar constants. The scheme under study has the form

$$\begin{aligned} &u^{n+1} = u^n - \tau L_x A u^n; \quad u^{n+2} = u^{n+1} - \tau L_y B u^{n+1}; \\ &u^{n+3} = u^{n+2} - \tau L_y B u^{n+2}; \quad u^{n+4} = u^{n+3} - \tau L_x A u^{n+3}. \end{aligned} \tag{58}$$

The operator L_x is again used at the $(n+5)$th time level, etc. The operators L_x and L_y approximate the operators $\partial/\partial x_1$ and $\partial/\partial x_2$, respectively. Eliminating the intermediate quantities u^{n+1}, u^{n+2}, and u^{n+3}, we obtain the difference equation of the form

$$u^{n+4} = Su^n, \tag{59}$$

where

$$S = (I - \tau L_x A)(I - \tau L_y B)^2(I - \tau L_x A), \tag{60}$$

I is the identity operator. Introducing the new dependent variables v^n, w^n, q^n by formulas $u^{n+1} = v^n,\ v^{n+1} = w^n,\ w^{n+1} = q^n$, we can write Eq. (59) as a system of difference equations,

$$q^{n+1} = Su^n, \quad u^{n+1} = v^n, \quad v^{n+1} = w^n, \quad w^{n+1} = q^n, \tag{61}$$

or in the vector matrix form:

$$\vec{u}^{\,n+1} = C\vec{u}^{\,n},$$

where

$$\vec{u}^{\,n} = \begin{pmatrix} q^n \\ u^n \\ v^n \\ w^n \end{pmatrix}, \quad C = \begin{pmatrix} 0 & S & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{pmatrix}.$$

The amplification matrix G obtained by the Fourier method has the form

$$G = \begin{pmatrix} 0 & a+ib & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{62}$$

where $a + ib$ is the Fourier symbol of operator S (60), a and b are the real functions of variables $\vec{\xi}$ and $\vec{\kappa} = (\kappa_1, \kappa_2)$, $\kappa_1 = A\tau/h_1$, and $\kappa_2 = B\tau/h_2$. It is easy to show that the characteristic equation of matrix G has the form $\lambda^4 = a + ib$. It follows from here that the von Neumann criterion has the form $a^2 + b^2 \leq 1$. In order to check the sufficiency of the von Neumann condition for the TVD scheme (58), we make use of the results of [10], namely, show that there exists a natural number N such that the condition for the normality of matrix G^N is satisfied:

$$(G^N)^* G^N - G^N (G^N)^* = 0. \tag{63}$$

The use of the computer algebra systems proved to be very efficient for the check-up of condition (63), because they enable us to perform a symbolic computation of the left-hand side of Eq. (63) on a computer. For this purpose we have used the computer algebra system *Mathematica 3.0* [19]. For the case of the amplification matrix G of the form (62) the program for the check-up of the satisfaction of condition (63) written in the language of the *Mathematica 3.0* system has the following form:

```
G={{0,a+I b,0,0},{0,0,1,0},{0,0,0,1},{1,0,0,0}};
kfin=5;
m=Length[G];Gzero=Table[0,{i,m},{j,m}];
Do[G0=MatrixPower[G,kj];
Print["G^",kj,"=",MatrixForm[G0]];
gs=ComplexExpand[Conjugate[Transpose[G0]]];
Print["(G^",kj,")*=",MatrixForm[gs]];
gd=G0.gs-gs.G0;
If[gd===Gzero,Print["Matrix G^",kj,"is normal"];
Break[]],{kj,kfin}];
```

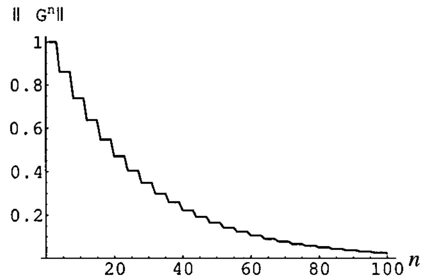

FIG. 5. The TVD scheme (58). The graph of $\|G^n\|$ as a function of the number n of time steps for $a = 0.7,\ b = 0.5$.

The input data for this program are the matrix G and the maximum power $N \geq 1$ ($N =$ `kfin` in our program) up to which the check-up of condition (63) is performed. The computation by the above program continues until condition (63) is satisfied at some N.

It turns out in the case of scheme (58) that condition (63) is satisfied at $N = 4$, because $G^4 = (a + ib)I$. Thus, in accordance with the theorem from [10] the von Neumann necessary stability condition is also sufficient for stability of scheme (58).

Note that G^4 is a diagonal matrix; therefore, the sufficiency of the von Neumann criterion also follows from Theorem 1.

The given matrix G of course has a sufficiently simple form, so that the check-up of condition (63) might be done without using the system *Mathematica*. However, in cases of sufficiently complex difference schemes approximating the systems of differential equations the dimension of the amplification matrix G may be large, and the entries of matrix G can be very complex functions of variables $\vec{\kappa}$ and $\vec{\xi}$. In such cases, the computation of the left-hand side of (63) for the increasing values N ($N = 1, 2, 3, \ldots$) proves to be a computationally intensive problem, which is practically not feasible using "manual" calculations, even for comparatively simple difference schemes.

We show in Fig. 5 the graph of $\|G^n\|$ as a function of n for $a = 0.7$, $b = 0.5$. The function $\|G^n\|$ has a similar form also for other values of a and b from the stability region. It can be seen that $\|G^n\|$ is a nonincreasing function of n, and the values of $\|G^n\|$ vary only at the values of n, which are the multiples of four.

4. UNIFORM STABILITY OF DIFFERENCE SCHEMES

The examples presented in the foregoing section show that it makes sense to introduce an additional characteristic of difference scheme, namely the concept of uniform stability.

DEFINITION 1. We shall say that the difference scheme is uniformly stable in some region of the parameters $\vec{\kappa} \in D$, if:

(1) the step operator $S(\vec{\kappa})$ is uniformly bounded:

$$\|S(\vec{\kappa})\| \leq M, \quad \vec{\kappa} \in D;$$

(2) there is a natural number n_0 such that at all values of the wave vector $\vec{\xi}$, the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ satisfies the condition:

$$\|G^{n+1}(\vec{\kappa}, \vec{\xi})\| \leq \|G^n(\vec{\kappa}, \vec{\xi})\| \quad \text{at } n \geq n_0, \vec{\kappa} \in D. \tag{64}$$

Remark. It is easy to show that a uniformly stable difference scheme is stable in the conventional sense. Let us take $\max_\xi$ of the both sides of inequality (64). We obtain the following inequality for the transition operator $C_{n,0}$:

$$\|C_{n+1,0}\| \leq \|C_{n,0}\| \quad \text{at } n \geq n_0.$$

Because the step operator S is bounded,

$$\|C_{n_0,0}\| \leq \|S\|^{n_0} \leq M^{n_0}.$$

Hence,

$$\|C_{n,0}\| \leq M^{n_0} \quad \text{for all } \; n \geq 1.$$

Let us now elucidate the question of the conditions which a difference scheme should satisfy in order to be uniformly stable. Consider a difference scheme with constant coefficients whose amplification matrix $G(\vec{\kappa}, \vec{\xi})$ is normal and the von Neumann condition $\max_\xi |\lambda_i(\vec{\kappa}, \vec{\xi})| \leq 1$, $\kappa \in D$ is satisfied. Such a difference scheme obviously will be stable. Because $G(\vec{\kappa}, \vec{\xi})$ is normal, it follows from relation

$$\|G(\vec{\kappa}, \vec{\xi})\| = \max_i |\lambda_i(\vec{\kappa}, \vec{\xi})| \tag{65}$$

and the von Neumann criterion that

$$\|G(\vec{\kappa}, \vec{\xi})\| = \max_i |\lambda_i(\vec{\kappa}, \vec{\xi})| \leq 1 \quad \text{for all } \; \xi; \quad \vec{\kappa} \in D. \tag{66}$$

Then we have the following relation for the powers of the matrix $G(\vec{\kappa}, \vec{\xi})$:

$$\begin{aligned} \|G^{n+1}(\vec{\kappa}, \vec{\xi})\| &\leq \|G(\vec{\kappa}, \vec{\xi})\| \, \|G^n(\vec{\kappa}, \vec{\xi})\| = \max_i |\lambda_i(\vec{\kappa}, \vec{\xi})| \, \|G^n(\vec{\kappa}, \vec{\xi})\| \\ &\leq \|G^n(\vec{\kappa}, \vec{\xi})\|, \quad \vec{\kappa} \in D, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{67}$$

That is, the norm of the nth degree of the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ is a nonincreasing function of the exponent n. Consequently, the difference schemes with the normal amplification matrix will be uniformly stable according to Definition 1.

Just the difference scheme (45) for the symmetrized acoustics equations, which was considered in Example 4, is such a difference scheme.

More general criteria for uniform stability of difference schemes are given by the following theorem.

THEOREM 3. *The difference scheme is uniformly stable in the region $\vec{\kappa} \in D$, if any of the following criteria for the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ is satisfied.*

Criterion 1. The eigenvalues $\gamma_i(\vec{\kappa}, \vec{\xi})$ of matrix $G^*(\vec{\kappa}, \vec{\xi})G(\vec{\kappa}, \vec{\xi})$ satisfy the condition

$$\max_{\vec{\xi}, i} |\gamma_i(\vec{\kappa}, \vec{\xi})| \leq 1, \quad \vec{\kappa} \in D.$$

Criterion 2.

1. Criterion 1 is satisfied in a subregion $\vec{\xi} \in \Omega_1(\vec{\kappa})$ of the periodicity region of the amplification matrix G.

2. The following conditions are satisfied in the complement of the set Ω_1 in the set Ω ($\vec{\xi} \in \Omega \backslash \Omega_1$):

(a) the matrix $G(\vec{\kappa}, \vec{\xi})$ is uniformly diagonalizable;

(b) there is a unique maximum eigenvalue $\lambda_1(\vec{\kappa}, \vec{\xi})$ of matrix G so that for all eigenvalues λ_i

$$|\lambda_i(\vec{\kappa}, \vec{\xi})| < |\lambda_1(\vec{\kappa}, \vec{\xi})|, \quad \textit{if } i \neq 1, \quad \max_{i \neq 1} \max_{\vec{\xi} \in \Omega \backslash \Omega_1} \frac{|\lambda_i(\vec{\kappa}, \vec{\xi})|}{|\lambda_1(\vec{\kappa}, \vec{\xi})|} < 1 - \delta_1, \quad \vec{\kappa} \in D; \tag{68}$$

(c)

$$\max_{\vec{\xi} \in \Omega \backslash \Omega_1} |\lambda_1(\vec{\kappa}, \vec{\xi})| < 1 - \delta_2, \quad \vec{\kappa} \in D, \tag{69}$$

where $\delta_1 > 0$, $\delta_2 > 0$ are some constants.

Remark. If the first criterion is satisfied then one can set $n_0 = 1$ in Definition 1.

Proof.

I. We make use of relation (41). Applying formulas (41) and

$$\|G^* G\| = \max_k \gamma_k \tag{70}$$

to the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ we obtain

$$\|G(\vec{\kappa}, \vec{\xi})\| = \left(\max_i \gamma_i(\vec{\kappa}, \vec{\xi})\right)^{1/2}. \tag{71}$$

Hence,

$$\|S(\vec{\kappa})\| = \max_{\vec{\xi}} \|G(\vec{\kappa}, \vec{\xi})\| = \max_{\vec{\xi}} \left[\max_i \gamma_i(\vec{\kappa}, \vec{\xi})\right]^{1/2} = \left[\max_{\vec{\xi}, i} \gamma_i(\vec{\kappa}, \vec{\xi})\right]^{1/2} \leq 1. \tag{72}$$

The condition of uniform stability (64) is now easily proved:

$$\|G^{n+1}(\vec{\kappa}, \vec{\xi})\| \leq \|S(\vec{\kappa})\| \, \|G^n(\vec{\kappa}, \vec{\xi})\| \leq \|G^n(\vec{\kappa}, \vec{\xi})\| \quad \text{for all } \; n \geq 1. \tag{73}$$

II. Because the matrix G is diagonalizable, $G = B^{-1} J B$, where J is a diagonal matrix. Then

$$G^n = \underbrace{(B^{-1} J B)(B^{-1} J B) \cdots (B^{-1} J B)}_{n \text{ factors}} = B^{-1} J^n B. \tag{74}$$

The matrix J^n is a diagonal matrix whose diagonal is filled with the nth powers of the eigenvalues of matrix G. Let k be the multiplicity of λ_1. Then one can present J^n in the form

$$J^n = \lambda_1^n \begin{pmatrix} \overbrace{1 \quad \quad \quad \quad}^{k \text{ columns}} & & & & & \\ & \ddots & & & & 0 \\ & & 1 & & & \\ & & & \varepsilon_2^n & & \\ & & & & \ddots & \\ & 0 & & & & \varepsilon_s^n \end{pmatrix}, \tag{75}$$

where $\varepsilon_i = \lambda_i/\lambda_1$. It follows from condition (68) that

$$|\varepsilon_i| < 1 - \delta_1. \tag{76}$$

The expression (75) for J^n may be written as the sum of two matrices:

$$J^n = \lambda_1^n \left[\begin{pmatrix} \overbrace{1 \quad \quad \quad \quad}^{k \text{ columns}} & & & & & \\ & \ddots & & & & 0 \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & 0 & & & & 0 \end{pmatrix} + \begin{pmatrix} \overbrace{0 \quad \quad \quad \quad}^{k \text{ columns}} & & & & & \\ & \ddots & & & & 0 \\ & & 0 & & & \\ & & & \varepsilon_2^n & & \\ & & & & \ddots & \\ & 0 & & & & \varepsilon_s^n \end{pmatrix} \right]$$

$$\equiv \lambda_1^n \left[T + O(\varepsilon^n) \right], \quad \varepsilon = 1 - \delta_1. \tag{77}$$

The nonzero entries of matrix $O(\varepsilon^n)$ tend to zero at $n \to \infty$ as ε^n. Substituting (77) in formula (74), we obtain

$$G^n = \lambda_1^n \left[B^{-1}TB + B^{-1}O(\varepsilon^n)B \right] \tag{78}$$

and

$$G^{n+1} = \lambda_1^{n+1}[B^{-1}TB + B^{-1}O(\varepsilon^{n+1})B]. \tag{79}$$

Let us estimate the norm of G^n from below and the norm of G^{n+1} from above:

$$\|G^n\| = |\lambda_1|^n \|B^{-1}TB + B^{-1}O(\varepsilon^n)B\| \geq |\lambda_1|^n [\|B^{-1}TB\| - \|B^{-1}O(\varepsilon^n)B\|].$$

It follows from here that

$$\|G^n\| \geq |\lambda_1|^n [\|B^{-1}TB\| - M_1\varepsilon^n], \tag{80}$$

where M_1 is a positive constant of the order of unity, which may be chosen independent of $\vec{\xi}$.

We obtain for $\|G^{n+1}\|$ that

$$\|G^{n+1}\| = |\lambda_1|^{n+1}\|B^{-1}TB + B^{-1}O(\varepsilon^{n+1})B\|$$
$$\leq |\lambda_1|^{n+1}[\|B^{-1}TB\| + \|B^{-1}O(\varepsilon^{n+1})B\|].$$

Therefore,

$$\|G^{n+1}\| \leq |\lambda_1|^{n+1}[\|B^{-1}TB\| + M_2\varepsilon^{n+1}]. \tag{81}$$

Let us now prove that the condition

$$\|G^{n+1}(\vec{\kappa}, \vec{\xi})\| \leq \|G^n(\vec{\kappa}, \vec{\xi})\|, \quad \kappa \in D \tag{82}$$

is satisfied beginning with a sufficiently large value of n for all values of vector $\vec{\xi} \in \Omega\backslash\Omega_1$. It follows from inequalities (80) and (81) that condition (82) is satisfied if the following inequality is valid:

$$|\lambda_1|^{n+1}[\|B^{-1}TB\| + M_2\varepsilon^{n+1}] \leq |\lambda_1|^n[\|B^{-1}TB\| - M_1\varepsilon^n]. \tag{83}$$

To prove inequality (83), let us divide both its sides by $|\lambda_1|^n$ and rewrite it as

$$(1 - |\lambda_1|)\|B^{-1}TB\| \geq (M_1 + M_2|\lambda_1|\varepsilon)\varepsilon^n. \tag{84}$$

In accordance with condition (69) of the theorem, we have the inequality $|\lambda_1(\vec{\kappa}, \vec{\xi})| < 1 - \delta_2$. Therefore, the inequality (84) is satisfied beginning with a sufficiently large value $n = n_0$, and consequently the condition (82) is satisfied.

The validity of inequality (82) for $\vec{\xi} \in \Omega_1$ and $n \geq 1$ follows from the first condition of Criterion 2.

The theorem is proved. ■

Let us compare the uniform stability concept with some known stability definitions. The concept of *strong stability* was proposed in [1]. The difference scheme is called strongly stable if there exists such a self-adjoint positive definite operator H so that

$$\|u^{n+1}\|_H \leq (1 + K\tau)\|u^n\|_H, \tag{85}$$

where u^n is the solution at the nth time level, and $\|\cdot\|_H$ is the norm determined by the operator H and is equivalent to the original L_2 norm,

$$\|u\|_H^2 = (u, Hu), \quad K_1^{-1}\|u\| \leq \|u\|_H \leq K_1\|u\|,$$

where K and K_1 are positive constants.

The definition of *uniform correctness* of the difference Cauchy problem, as given in [3], is

$$\|S\| \leq 1 + M\tau, \quad M > 0, \tag{86}$$

where S is the step operator. It is easy to prove that the strong stability of a difference scheme follows from its uniform correctness:

$$\|u^{n+1}\| \leq \|S\|\,\|u^n\| \leq (1 + M\tau)\|u^n\|. \tag{87}$$

Condition (87) coincides with condition (85) for $H \equiv I$.

It is obvious that on the one hand the uniform stability condition (64) imposes weaker limitations on the difference scheme than the condition of strong stability (85) and the condition of uniform correctness (86), because condition (64) admits the growth of the norm of the transition operator $C_{n,0}$ in the initial interval of time steps n. This difference plays an important role, because the norms of the transition operators for a number of practically important difference schemes demonstrate just such a behavior: the growth in the initial interval $1 \le n \le n_0$ and the decay at $n \ge n_0$.

On the other hand, for $n \ge n_0$ the uniform stability condition imposes more severe limitations, because it requires the satisfaction of condition (64) at all $\vec{\xi}$. In contrast, the condition

$$\|C_{n+1,0}\| \le \|C_{n,0}\|$$

does not eliminate the violation of condition (64) at some values of $\vec{\xi}$, which may lead to spurious oscillations with a finite wavelength.

From the previously known results, we note also the stability conditions for difference schemes with variable coefficients obtained in [11]. The first of the conditions has the form

$$G^*(\vec{\kappa}, \vec{\xi})G(\vec{\kappa}, \vec{\xi}) \le I \quad \text{for all } \vec{\kappa},\ \vec{\xi}, \tag{88}$$

where $G(\vec{\kappa}, \vec{\xi})$ is the amplification matrix, I is the identity matrix, and $\vec{x}$ is the vector of spatial variables. For difference schemes with constant coefficients, condition (88) means that

$$\|G(\vec{\kappa}, \vec{\xi})\|^2 \le 1 \quad \text{for all } \vec{\xi};$$

hence, $\|S\| \le 1$. Therefore, the difference scheme is uniformly stable at $n_0 = 1$ when (88) is satisfied, and the first criterion of Theorem 3 gives a constructive method for checking condition (88).

Let us illustrate the application of Theorem 3 for a number of the specific difference schemes. Consider the TVD scheme from Example 5. It is easy to show that the eigenvalues of matrix G^*G for this scheme satisfy the equation

$$\max_i |\gamma_i| = \max(1, a^2 + b^2).$$

Consequently, under the satisfaction of the von Neumann condition $a^2 + b^2 \le 1$, the first criterion of Theorem 3 is satisfied. Therefore, this scheme is uniformly stable.

EXAMPLE 6. Let us consider the two-cycle MacCormack scheme [20] for the advection equation (57). The scheme under study has the form

$$u^{n+1} = L_1 u^n, \quad u^{n+2} = L_2 u^{n+1}. \tag{89}$$

The step operator L_1 is again applied at the $(n+3)$rd time level, and the step operator L_2 is applied at the $(n+4)$th time level; that is, the operators L_1 and L_2 are applied cyclically with period 2τ. The forms of the operators L_1 and L_2 are presented in [20–22]. Substituting the expression for u^{n+1} from the first equation of (89) in the second equation, we obtain a three-level difference scheme of the form

$$u^{n+2} = L_2 L_1 u^n. \tag{90}$$

The amplification matrix G corresponding to (90) has the form

$$G = \begin{pmatrix} 0 & 1 \\ a + ib & 0 \end{pmatrix}, \tag{91}$$

where a and b are the real functions of the variables $\vec{\xi}$ and $\vec{\kappa} = (\kappa_1, \kappa_2)$; $\kappa_1 = A\tau/h_1$, $\kappa_2 = B\tau/h_2$. In view of the bulky form of the expressions for a and b, we do not present them here (the explicit form of a and b may be found in [22]).

It was shown in [21, 23] with the aid of a symbolic-numerical method that the necessary condition for stability of scheme (90) has the form

$$\kappa_1^{2/3} + \kappa_2^{2/3} \leq 1. \tag{92}$$

A strict proof of condition (92) was given in the works [24, 25]. It is easy to check that matrix G (91) is not a normal matrix. However, the matrix G^2 is normal, because

$$G^2 = (a + ib)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the two-cycle MacCormack scheme satisfies the conditions of the theorem from [10]. Therefore, the von Neumann condition (92) is sufficient for stability, and the scheme is stable in the overall region (92) of the parameters $\kappa_1,\ \kappa_2$.

Let us prove the uniform stability of this scheme. A direct computation gives

$$G^*G = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We obtain from here

$$\|G\|^2 = \max(1, a^2 + b^2) = 1$$

under the satisfaction of the von Neumann condition. Although the amplification matrix (91) is not normal, the difference scheme (90) is nevertheless uniformly stable with $n_0 = 1$ under the satisfaction of the von Neumann condition.

EXAMPLE 7. Let us again consider the difference scheme (39) for the wave equation. Let us check the satisfaction of criterion 1 of Theorem 3. The eigenvalues of matrix G^*G are easily found and have the form

$$\gamma_1 = \frac{1}{2}(2 + a^4 + a^2\sqrt{4 + a^4}), \quad \gamma_2 = \frac{1}{2}(2 + a^4 - a^2\sqrt{4 + a^4})),$$

where $a = 2\kappa \sin(\xi/2)$. This scheme was shown above to be stable at $0 < \kappa < 1$. However, at $0 < \xi < 2\pi$ the eigenvalue γ_1 is larger than unity and, hence, the first criterion of Theorem 3 is not satisfied.

Consider the second criterion. It is easy to show that at $0 < \kappa < 1$ the amplification matrix G is diagonalizable, but at all ξ it has two complex conjugate eigenvalues

$$\lambda_{1,2} = \frac{2 - a^2}{2} \pm \frac{ia}{2}\sqrt{4 - a^2}$$

($|\lambda_1| = |\lambda_2| = 1$). Thus, the second criterion of Theorem 3 is also not satisfied. The numerical results presented in Figs. 2a–2d also show that this difference scheme is not uniformly stable at least for $n \leq 100$.

Let us prove that difference scheme (39) is not uniformly stable at any n_0. The amplification matrix G may be presented in the form

$$G = RTR^{-1}, \tag{93}$$

where

$$R = \begin{pmatrix} -\frac{1}{2}\sqrt{4-a^2} - \frac{1}{2}ia & \frac{1}{2}\sqrt{4-a^2} - \frac{1}{2}ia \\ 1 & 1 \end{pmatrix},$$

and

$$T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \quad \lambda_1 = \frac{1}{2}(2 - a^2 + i\sqrt{4-a^2}).$$

Write the expression for λ_1 as

$$\lambda_1 = e^{i\varphi_1}, \quad \varphi_1 = \arctan\frac{\sqrt{4-a^2}}{2-a^2}.$$

Then we obtain the expression for G^n,

$$G^n = RT^nR^{-1} \tag{94}$$

$$T^n = \begin{pmatrix} e^{in\varphi_1} & 0 \\ 0 & e^{-in\varphi_1} \end{pmatrix},$$

and G^{n+1} may be presented as

$$G^{n+1} = RT^nR^{-1}G. \tag{95}$$

Because φ_1 is a continuous function of ξ_1, the set of rational numbers will be everywhere dense on the set of the φ_1 values. Choose such a $\xi_0 \in (0, 2\pi)$ that $\varphi_1(\xi_0) = 2\pi m_1/m_2$, m_1, and m_2 are the natural numbers. Then at $n = km_2$, we obtain that $T^n = I$, and

$$\|G^n(\xi_0)\| = 1 \tag{96}$$

and

$$\|G^{n+1}(\xi_0)\| = \|G(\xi_0)\| > 1. \tag{97}$$

Hence, $\|G^{n+1}(\xi_0)\| > \|G^n(\xi_0)\|$ at arbitrarily large $n = km_2$; therefore, condition (64) is not satisfied.

EXAMPLE 8. Let us again consider Fletcher's scheme (34). Let us show that this scheme is not uniformly stable. Numerical computations have shown that for all values $\kappa \leq 1$,

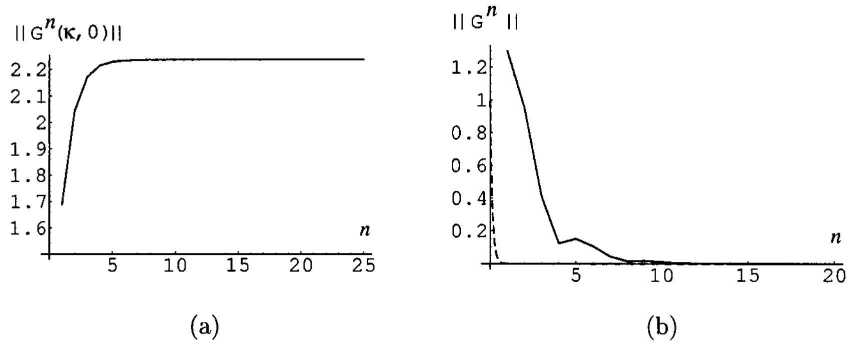

FIG. 6. Fletcher's scheme (34). The graphs of $\|G^n\|$ as functions of the number n of time steps: (a) $\xi = 0$; (b) $\kappa = 0.8$, $\xi = \pi$; (- - -) the graph of the function $e^{-\kappa\xi^2 n}$ at $\kappa = 0.8$, $\xi = \pi$.

the first criterion of Theorem 3 is not satisfied, because $\max_{\xi,i} \gamma_i(\kappa, \xi) > 1$, γ_i are the eigenvalues of the matrix G^*G. Consider the second criterion of Theorem 3.

1. For the values of the parameter $\kappa < \frac{1}{4}(2 - \sqrt{3})$, there exist different positive eigenvalues (38). However, at $\xi = 0$ the maximum eigenvalue $\lambda_1(\kappa, 0)$ becomes to be equal to unity, and condition (69) is not satisfied.

The numerical results presented in Fig. 6a show that at $\xi = 0$ the norm $\|G^n(\kappa, 0)\|$, as a function of n, is an increasing function of n and tends asymptotically to certain finite limit: $\lim_{n\to\infty} \|G^n(\kappa, 0)\| = C_0$. Consequently, the scheme (34) is not uniformly stable for $\kappa < (1/4)(2 - \sqrt{3})$.

2. In the region $(1/4)(2 - \sqrt{3}) < \kappa \leq 1$, there exists an interval of values of the variable $\xi \in (\xi_1, \xi_2)$ (see Fig. 7b) in which the eigenvalues become complex conjugate and $|\lambda_1| = |\lambda_2|$. Therefore, condition (69) of Theorem 3 is not satisfied. The numerical computations show that there exists a sequence of the time step numbers $n_1, n_2, \ldots,$ at which the uniform stability condition (64) is violated at $\xi \in (\xi_1, \xi_2)$ (see Fig. 6b). The absence of uniform stability at $\xi \in (\xi_1, \xi_2)$ can be proved strictly similarly to the proof made in Example 7.

One can illustrate at this example the practical importance of the uniform stability concept. As is known [3], the solution of the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \\ u(x, 0) &= u_0(x), \quad -\infty < x < \infty, \end{aligned} \tag{98}$$

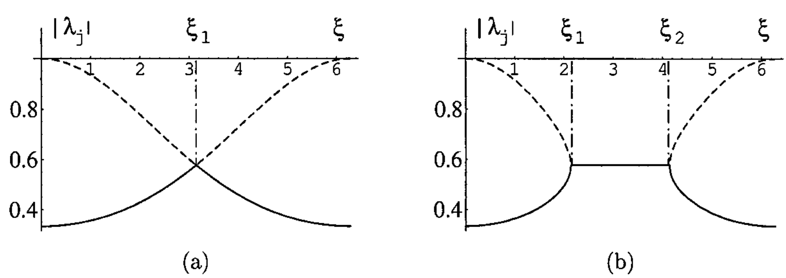

FIG. 7. Fletcher's scheme (34). The absolute values of eigenvalues as functions of ξ: (- - -) $|\lambda_1(\xi)|$, (——) $|\lambda_2(\xi)|$; (a) $\kappa = \frac{1}{4}(2 - 3^{1/2})$; (b) $\kappa = \frac{1.3}{4}(2 - 3^{1/2})$.

where $u_0(x)$ is a given function, may be presented in the form

$$u(x, t) = S(t, 0)u_0(x), \tag{99}$$

where $S(t, 0)$ is the transition operator whose form may be obtained with the aid of the Fourier transform of (33). Because the system of difference equations (35) contains two equations, it is necessary for the computation of $\|S(t, 0)\|$ to use also a system of two equations while studying the properties of the corresponding transition operator $S(t, 0)$. The first equation of system (35) introduces a shift in time: $v(x, t) = u(x, t + \tau)$. Therefore, we consider instead of a single equation (33) the system

$$\begin{cases} v(x, t) = u(x, t + \tau); \\ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}. \end{cases} \tag{100}$$

The Fourier transform of system (100) leads to the expression for $\|S(t, 0)\|_{L_2}$ [3],

$$\|S(t, 0)\|_{L_2} = \sup_k \|S(k, t, 0)\| = \sup_k e^{-\nu k^2 t} = 1, \tag{101}$$

where k is a real wavenumber, $-\infty < k < \infty$. The expression for the matrix $S(k, t, 0)$ is obtained as follows. Let $C_m(k, t)$ be the Fourier coefficients of functions $u(x, t)$, $v(x, t)$, $m = 1, 2$. Then

$$C_1(k, t) = e^{-\nu k^2 t} C_1(k, 0), \quad C_2(k, t) = e^{-\nu k^2 t} C_2(k, 0), \tag{102}$$

where $C_2(k, 0) = C_1(k, 0)e^{-\nu k^2 \tau}$ with regard for the definition of the function $v(x, t)$. We obtain from (102)

$$S(k, t, 0) = \mathrm{diag}\left(e^{-\nu k^2 t}, e^{-\nu k^2 t}\right). \tag{103}$$

Because the matrix S (103) is a normal matrix, it is obvious that

$$\|S(k, t, 0)\| = e^{-\nu k^2 t}, \tag{104}$$

which is reflected in formula (101).

Thus, by virtue of the construction of the transition operator $S(t, 0)$ (see (99)) we can see that the quantity $\|S(k, t, 0)\|$ is a continuous analog of the quantity $\|G^n\|$. Therefore, in the case of very accurate difference scheme, the quantities $\|S(k, t, 0)\|$ and $\|G^n\|$ should be close to each other. Assuming $t = n\tau$ in (104), we can rewrite formula (104) as

$$\|S(k, n\tau, 0)\| = e^{-\kappa \xi^2 n}, \tag{105}$$

where $\kappa = \nu\tau/(h^2)$, $\xi = kh$. It follows from (105) that the quantity $\|S(k, n\tau, 0)\|$ is a monotone decreasing function of n, and $\|S(k, 0, 0)\| = 1$. The graph of the function $e^{-\kappa \xi^2 n}$ is shown in Fig. 6b. It can be seen that the behavior of the quantity $\|G^n\|$ differs significantly from the behavior of the quantity $\|S(k, n\tau, 0)\|$. This difference is striking at $\xi = 0$, since $\|S(0, t, 0)\| \equiv 1$, but it is seen from Fig. 6a that $\|G^n\|$ tends asymptotically to a constant exceeding two.

We have shown that scheme (34) is not uniformly stable. The behavior of $\|G^n(\kappa, \xi)\|$ is nevertheless qualitatively different in two intervals of the parameter κ considered above.

At $\frac{1}{4}(2-\sqrt{3}) \le \kappa \le 1$, the scheme oscillations have the finite wavelengths ($\xi \in (\xi_1, \xi_2)$). If $\kappa < \frac{1}{4}(2-\sqrt{3})$, then the violation of the uniform stability condition occurs only in small neighborhoods of the values $\xi = 0$ and $\xi = 2\pi$, and the corresponding scheme oscillations have the large wavelengths. Because the disturbances with finite wavelengths play the most important role in many physical processes, the long-wave scheme oscillations may prove to be insignificant.

Therefore, it makes sense to introduce the concept of a locally uniform stability.

DEFINITION 2. A stable difference scheme is called locally uniformly stable if at arbitrarily small $\delta > 0$ the amplification matrix $G(\vec{\kappa}, \vec{\xi})$ satisfies the condition

$$\|G^{n+1}(\vec{\kappa}, \vec{\xi})\| \le \|G^n(\vec{\kappa}, \vec{\xi})\| \tag{106}$$

at $\vec{\xi} \in [\delta, 2\pi - \delta]^L$ and $n \ge n_0(\delta)$, where

$$[\delta, 2\pi - \delta]^L = \overbrace{[\delta, 2\pi - \delta] \times \cdots \times [\delta, 2\pi - \delta]}^{L},$$

L is the dimension of vector $\vec{\xi}$, and $n_0(\delta)$ may go to ∞ as $\delta \to 0$.

The conditions for which a stable difference scheme is locally uniformly stable are given by the following theorem.

THEOREM 4. *A stable difference scheme is locally uniformly stable in the region $\vec{\kappa} \in D$ if at all values of the wave vector $\vec{\xi} \in [\delta, 2\pi - \delta]^L$, where $\delta > 0$ is an arbitrarily small quantity, the following conditions are satisfied*:

(1) the matrix $G(\vec{\kappa}, \vec{\xi})$ is diagonalizable;

(2) there exists the unique maximum eigenvalue $\lambda_1(\vec{\kappa}, \vec{\xi})$ of matrix $G(\vec{\kappa}, \vec{\xi})$:

$$|\lambda_i(\vec{\kappa}, \vec{\xi})| < |\lambda_1(\vec{\kappa}, \vec{\xi})|, \quad \text{if } i \ne 1;$$

(3) $|\lambda_1(\vec{\kappa}, \vec{\xi})| < 1$.

The proof of Theorem 4 is quite similar to the proof of the second criterion of Theorem 3.

Let us return to Example 8. It is easy to see from formulas (38) that for $0 < \kappa < \frac{1}{4}(2-\sqrt{3})$ the amplification matrix has two different real eigenvalues, and $\lambda_1 = 1$ for $\xi = 0,\ 2\pi$ and $\lambda_1 < 1$ for $\xi \in [\delta, 2\pi - \delta]$, where $\delta > 0$ is arbitrarily small. Therefore, the conditions of Theorem 4 are satisfied, and scheme (34) is locally uniformly stable in the region $0 < \kappa < \frac{1}{4}(2-\sqrt{3})$.

5. INVESTIGATION OF UNIFORM STABILITY OF DIFFERENCE SCHEMES FOR THE EULER EQUATIONS

We have investigated above the uniform stability of a number of difference schemes approximating the scalar partial differential equations of hyperbolic and parabolic type. However, difference schemes approximating the Euler equations for the compressible inviscid gas are of more interest. This is related to the fact that one has to execute a large number of time steps in a number of applied problems in order to obtain a complete solution of the problem. The examples of such problems are

- stationary problems solved by the pseudo-unsteady method;
- the problems of long-term weather forecast;
- the problems of hydrodynamic stability.

At the numerical solution of these problems it is very important to ensure that the difference scheme first remains stable at large times and that the eigenoscillations or the errors introduced by the difference scheme itself are sufficiently small during the overall calculation and do not introduce distortions in the problem solution.

In this work, we restrict ourselves to the consideration of difference schemes approximating the Euler equations governing the one-dimensional flow of an inviscid compressible gas. The divergence form of these equations is

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0, \tag{107}$$

where

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad f(w) = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ pu + \rho u E \end{pmatrix}. \tag{108}$$

Here x is the Cartesian spatial coordinate, t is the time, ρ is the gas density, u is the velocity, p is the pressure, E is the specific total energy, $E = \varepsilon + \frac{u^2}{2}$, and ε is the specific internal energy. We will assume that the equation of state $p = F(\rho, \varepsilon)$ is used to complete the system (107), (108), where F is a given function.

The nondivergence form of the Euler equations (107)–(108) is

$$\frac{\partial w}{\partial t} + A\frac{\partial w}{\partial x} = 0, \tag{109}$$

where $A = A(w) = \partial f(w)/\partial w$. Because we will use the Fourier method for the stability analysis of difference schemes approximating the system (107), (108), we will have to linearize the difference equations. The linearized difference equations indeed approximate the system (109), where the matrix A is assumed to be constant. Therefore, we will write down the difference schemes directly for system (109). We at first consider an example of a first-order difference scheme: the Lax–Friedrichs scheme [26]:

$$w_j^{n+1} = \frac{1}{2}\left(w_{j-1}^n + w_{j+1}^n\right) - \frac{\tau}{2h}A\left(w_{j+1}^n - w_{j-1}^n\right). \tag{110}$$

The amplification matrix G of scheme (110) has the form

$$G = \cos\xi I - \left(\frac{\tau}{h} i \sin\xi\right)A, \tag{111}$$

where $\xi = kh$, and k is the real wavenumber. As is known [27], the von Neumann stability condition of scheme (110) has the form

$$(|u| + c)\tau/h \leq 1. \tag{112}$$

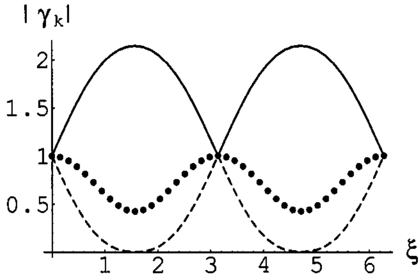

FIG. 8. The Lax–Friedrichs scheme (110). The modules of eigenvalues γ_i as functions of ξ at $\kappa_1 = 0.4,\ \kappa_2 = 0.5$: (– – –) $|\gamma_1(\vec{\kappa}, \xi)|$; (· · ·) $|\gamma_2(\vec{\kappa}, \xi)|$; (——) $|\gamma_3(\vec{\kappa}, \xi)|$.

Let T be the matrix of eigenvectors corresponding to matrix A. Then, as is known, the matrix A may be reduced to the diagonal form

$$A_0 = TAT^{-1} = \operatorname{diag}(\mu_1, \mu_2, \mu_3), \tag{113}$$

with the aid of similarity transformation, where μ_1, μ_2, μ_3 are the eigenvalues of matrix $A = \partial f(w)/\partial w$. As is known [1, 27],

$$\mu_1 = u, \quad \mu_2 = u + c, \quad \mu_3 = u - c, \tag{114}$$

where c is the sound velocity. Thus, the matrix G (111) belongs to the class of diagonalizable amplification matrices. It is well known for such matrices that the von Neumann necessary stability condition is also sufficient for difference scheme stability (see also Theorem 1 above).

Let us investigate the question of the uniform stability of scheme (110). A direct computation of the eigenvalues γ_i of the matrix G^*G has shown that the first criterion of Theorem 3 is not satisfied (see Fig. 8).

Consider the second criterion. We obtain with regard for (110) the expressions for the eigenvalues $\lambda_i(\vec{\kappa}, \vec{\xi})$ of matrix G,

$$\begin{aligned}
\lambda_1(\vec{\kappa}, \vec{\xi}) &= \cos\xi - i(\kappa_1 + \kappa_2)\sin\xi, \\
\lambda_2(\vec{\kappa}, \vec{\xi}) &= \cos\xi - i\kappa_2\sin\xi, \\
\lambda_3(\vec{\kappa}, \vec{\xi}) &= \cos\xi - i(\kappa_2 - \kappa_1)\sin\xi,
\end{aligned} \tag{115}$$

where

$$\vec{\kappa} = (\kappa_1, \kappa_2), \quad \kappa_1 = \frac{c\tau}{h}, \quad \kappa_2 = \frac{u\tau}{h}. \tag{116}$$

We present in Fig. 9 the graph of $|\lambda_i(\vec{\kappa}, \vec{\xi})|$ $(i = 1, 2, 3)$ for the case $u \neq 0$, $u \neq c$. It is seen that the $|\lambda_i(\vec{\kappa}, \vec{\xi})|$ coincide and are equal to unity at points $\xi = 0,\ \xi = \pi,\ \xi = 2\pi$. It can be seen from Fig. 8 that

$$\max_{\xi \in [0,\delta]} |\gamma_i(\vec{\kappa}, \vec{\xi})| > 1$$

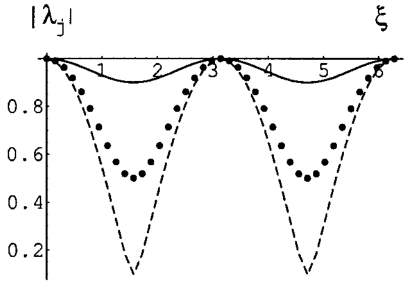

FIG. 9. The Lax–Friedrichs scheme (110). The modules of eigenvalues as functions of ξ at $\kappa_1 = 0.4$, $\kappa_2 = 0.5$: (– – –) $|\lambda_1(\vec{\kappa}, \xi)|$; (· · ·) $|\lambda_2(\vec{\kappa}, \xi)|$; (——) $|\lambda_3(\vec{\kappa}, \xi)|$.

in the neighborhood of point $\xi = 0$: $\xi \in [0, \delta]$, where $\delta > 0$ is arbitrarily small. Therefore, criterion 2 of Theorem 3 is also not satisfied.

The conditions of Theorem 4 are also not satisfied, because the $|\lambda_i(\vec{\kappa}, \vec{\xi})|$ coincide and are equal to unity at point $\xi = \pi$. Therefore, it is to be expected that scheme (110) is neither uniformly stable nor locally uniformly stable. The numerical computations of $\|G^n\|$ as a function of n confirm this conclusion. Note that $\|G^n\|$ depends not only on the nondimensional parameters (116) but also on the dimensional parameter $r = \tau/h$. In the present work, the basic numerical results are presented for $r = 1$, because at different values of r the picture does not change qualitatively.

We present in Fig. 10 the graphs of $\|G^n\|$ at different values of ξ from a small neighborhood of π. It may be seen that as ξ approaches the value $\xi = \pi$ the regime of a uniform decay of $\|G^n\|$ begins at larger values $n = n_1$. The numerical computations show that $n_1(\xi) \to \infty$ at $\xi \to \pi$.

Thus, in the case where $u \neq 0$, $u \neq c$, the scheme (110) is not locally uniformly stable. In the numerical computations using this scheme, the spurious oscillations may arise with the wave numbers that are close to π/h.

The case of transonic flow $u = c$ ($\kappa_2 = \kappa_1$) does not differ qualitatively from the foregoing case. The condition of a locally uniform stability is violated as before in the neighborhood of point $\xi = \pi$.

As can be seen from (115), in the case $u = 0$ ($\kappa_2 = 0$) there exist two complex conjugate eigenvalues λ_1 and λ_3, which have the same maximum modulus. The conditions of

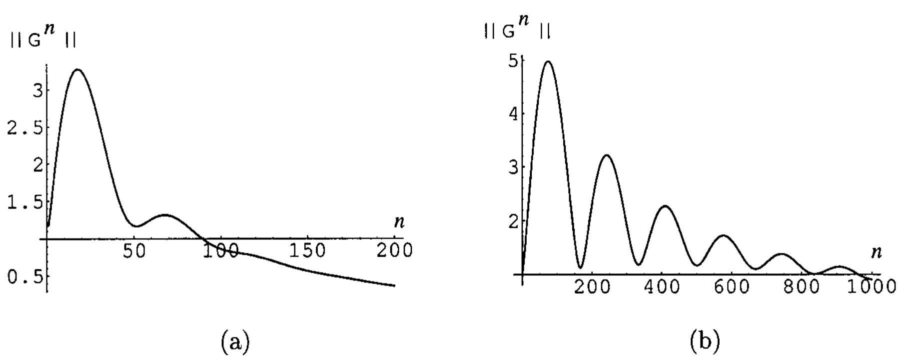

FIG. 10. The Lax–Friedrichs scheme (110). The graphs of $\|G^n\|$ as functions of the number n of time steps: (a) $\xi = 1.1\pi$; (b) $\xi = 1.03\pi$.

Theorem 4 are violated at all values of ξ. Therefore, the scheme can oscillate with any finite wave length.

Now consider an example of a second-order Lax–Wendroff difference scheme for system (109) [1, 27, 28]

$$w_j^{n+1} = w_j^n - \frac{\tau}{2h} A\left(w_{j+1}^n - w_{j-1}^n\right) + \frac{\tau^2}{2h^2} A^2\left(w_{j+1}^n - 2w_j^n + w_{j-1}^n\right). \tag{117}$$

The amplification matrix G for scheme (117) has the form

$$G = I - \left(\frac{\tau}{h} i \sin\xi\right) A + \frac{\tau^2}{h^2}(\cos\xi - 1)A^2. \tag{118}$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of matrix (118) are given by

$$\begin{aligned} \lambda_1(\vec{\kappa}, \xi) &= 1 - i\kappa_2 \sin\xi + (\cos\xi - 1)\kappa_2^2, \\ \lambda_2(\vec{\kappa}, \xi) &= 1 - i(\kappa_1 + \kappa_2)\sin\xi + (\cos\xi - 1)(\kappa_1 + \kappa_2)^2, \\ \lambda_3(\vec{\kappa}, \xi) &= 1 - i(\kappa_2 - \kappa_1)\sin\xi + (\cos\xi - 1)(\kappa_2 - \kappa_1)^2, \end{aligned} \tag{119}$$

where the vector $\vec{\kappa}$ is determined in accordance with (116). The von Neumann necessary stability condition also has the form (112), or in terms of κ_1, κ_2,

$$|\kappa_2| + \kappa_1 \leq 1. \tag{120}$$

The matrix G for this scheme is also diagonalizable; therefore, the difference scheme (117) is stable in the region (120).

A direct calculation of the eigenvalues γ_i of matrix G^*G shows that $\max|\gamma_i| > 1$ for all values $0 < \xi < 2\pi$; consequently, the first criterion of Theorem 3 is not satisfied. Let us show that criterion 2 of Theorem 3 is also not satisfied. Let us study the behavior of the eigenvalues $\lambda_i(\vec{\kappa}, \xi)$. We show in Fig. 11a the graphs of the eigenvalues at $u \neq 0, u \neq c$. It can be seen that the eigenvalues coincide at $\xi = 0, \xi = 2\pi$: $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Because $\max|\gamma_i| > 1$ in any arbitrarily small neighborhoods of points $\xi = 0,\ 2\pi$, the first condition

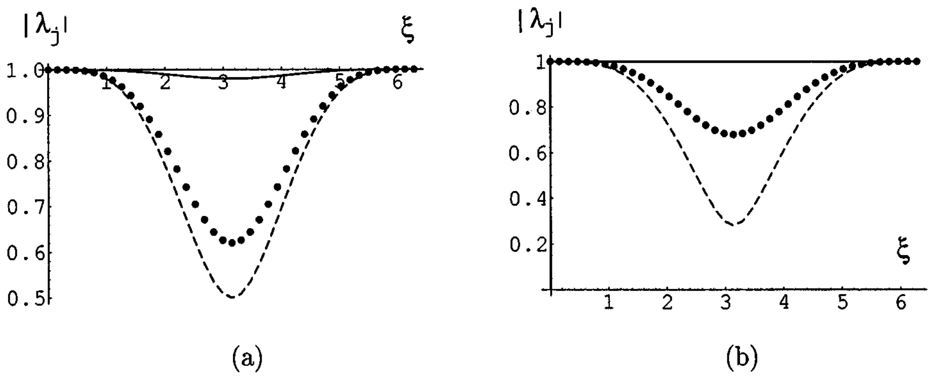

FIG. 11. The Lax–Wendroff scheme (117). The modules of eigenvalues as functions of ξ: (a) $\kappa_1 = 0.4,\ \kappa_2 = 0.5$; (b) $\kappa_1 = \kappa_2 = 0.4$. (– – –) $|\lambda_1(\vec{\kappa}, \xi)|$; (· · ·) $|\lambda_2(\vec{\kappa}, \xi)|$; (——) $|\lambda_3(\vec{\kappa}, \xi)|$.

of Criterion 2 is not satisfied. Therefore, it is to be expected that scheme (117) is not uniformly stable.

Because the eigenvalues $\lambda_i(\vec{\kappa}, \vec{\xi})$ of matrix G have different absolute values for all $\xi \in [\delta, 2\pi - \delta]$, where $\delta > 0$ is arbitrarily small, the conditions of Theorem 4 are satisfied, and scheme (117) is locally uniformly stable at $u \neq 0$, $u \neq c$.

Consider the case $u = c$ (the transonic flow regime). The graphs of the eigenvalues are presented in Fig. 11b. It is seen from (119) that the maximum eigenvalue $\lambda_3 = 1$ in the overall interval $\xi \in [0, 2\pi]$. Consequently, at the parameter values $\kappa_2 = \kappa_1$ the scheme (117) is no longer locally uniformly stable, and the appearance of spurious oscillations with any wavelength becomes possible.

The same picture is observed at $u = 0$ ($\kappa_2 = 0$). The eigenvalue λ_1, which is equal to unity for all ξ, now becomes the maximum one.

Summarizing the obtained results, one can assert that scheme (117) is locally uniformly stable in the region

$$|\kappa_2| + \kappa_1 \leq 1, \quad \kappa_2 \neq 0, \kappa_2 \neq \kappa_1, \tag{121}$$

and it is stable in conventional sense at $\kappa_2 = 0$ or $\kappa_2 = \kappa_1$ if $|\kappa_2| + \kappa_1 \leq 1$.

A comparative analysis of the Lax–Friedrichs scheme (110) and the Lax–Wendroff scheme (117) shows that scheme (117) is better on the whole than scheme (110), because it is locally uniformly stable almost in the total stability region (121). At $u = c$ the situation, however, changes, because scheme (110) allows the generation of oscillations only with the wave numbers close to π/h, whereas scheme (117) may give rise to spurious oscillations with any wavelength. An equally poor behavior is demonstrated by both schemes (110) and (117) in the case of stagnation flow, when $u \approx 0$. In this case, both schemes generate oscillations whose period depends on κ_1 and ξ. Because the gas flow is certainly subsonic in the stagnation zone itself and in its neighborhood, the scheme oscillations propagate along the characteristics inside the spatial computational region and can distort the solution in the overall region at sufficiently large values of n, although the computational process will remain stable. Therefore, it is not desirable to apply schemes (110) and (117) to those problems of gas dynamics, where there are the subregions of stagnation.

A locally uniformly stable scheme is of course better than a simply stable scheme. The value n_0 in Definition 1 characterizing the passage to the regime of uniform stability may, however, be sufficiently large even at not very small values of ξ, as the analysis of scheme (117) has shown. We present in Fig. 12 the graphs of the quantity $\|G^n\|$ at $\xi = \frac{\pi}{3}$. It can be

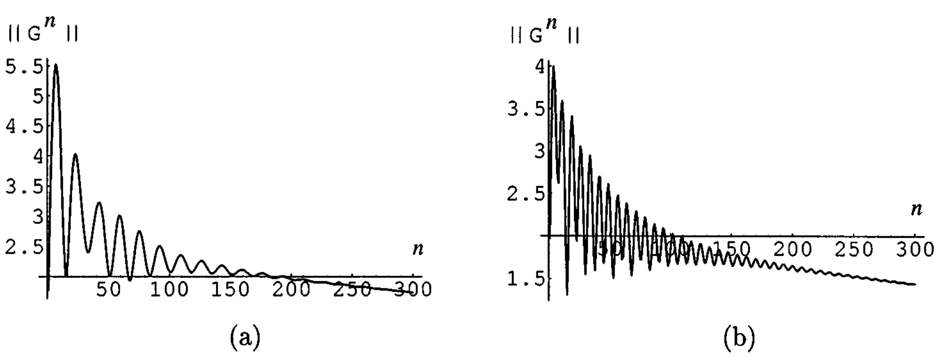

FIG. 12. The Lax–Wendroff scheme (117). The graphs of $\|G^n\|$ as functions of the number n of time steps at $\kappa_1 = 0.4$, $\kappa_2 = 0.5$, $\xi = \pi/3$: (a) $r = 1$; (b) $r = 0.5$.

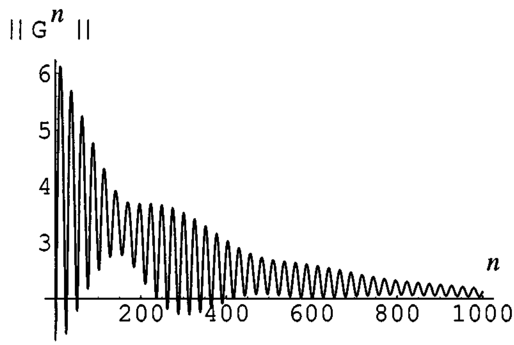

FIG. 13. The Lax–Wendroff scheme (117). The graph of $\|G^n\|$ as a function of the number n of time steps at $\kappa_1 = 0.4,\ \kappa_2 = 0.5,\ r = 1,\ \xi = \pi/5$.

seen from Fig. 12a that one must set $n_0 \approx 230$ in Definition 1 at $r = 1$. At $r = 0.5$ one has to increase the value of n_0; see Fig. 12b. It is seen from Fig. 13 that at a further reduction of ξ there is no passage of $\|G^n\|$ to the uniform stability regime even at $n = 1000$. Therefore, it is desirable to have a difference scheme possessing uniform stability.

Consider the question of how one could improve the Lax–Friedrichs scheme and the Lax–Wendroff scheme and ensure their uniform stability at all ξ. The artificial dissipation terms are known to smooth out the numerical solution. Therefore, one could expect that their use in the difference scheme would smooth out also the graph of $\|G^n\|$, damping the oscillations of this quantity.

Following [29] let us introduce the artificial dissipator D in the system (109)

$$\frac{\partial w}{\partial t} + A\frac{\partial w}{\partial x} = D,$$

where

$$D = \kappa^{(2)}\frac{h^2}{\tau}\frac{\partial^2 w}{\partial x^2} - \kappa^{(4)}\frac{h^4}{\tau}\frac{\partial^4 w}{\partial x^4}. \tag{122}$$

Here $\kappa^{(2)} \geq 0$ and $\kappa^{(4)} \geq 0$ are the artificial dissipation coefficients. The Lax–Wendroff scheme (117) is then modified as

$$w_j^{n+1} = w_j^n - \frac{\tau}{2h}A\left(w_{j+1}^n - w_{j-1}^n\right) + \frac{\tau^2}{2h^2}A^2\left(w_{j+1}^n - 2w_j^n + w_{j-1}^n\right) + \tau D_j^n. \tag{123}$$

Assume that the difference approximation D_j^n of the artificial dissipator (122) has the form

$$D_j^n = \sum_{k=-q_1}^{q_2} c_k w_{j+k}^n, \tag{124}$$

where $q_1 \geq 0,\ q_2 \geq 0,\ q_1 + q_2 > 0$, and the coefficients c_k depend on $\kappa^{(2)}$ and $\kappa^{(4)}$. To ensure vanishing of the quantity D_j^n on a constant grid function u^n, we require satisfaction of the equality

$$\sum_{k=-q_1}^{q_2} c_k = 0\,. \tag{125}$$

The amplification matrix G of scheme (123), (124) has the form

$$G = I - \left(\frac{\tau}{h} i \sin\xi\right) A + \frac{\tau^2}{h^2}(\cos\xi - 1)A^2 + \tau \sum_{k=-q_1}^{q_2} c_k e^{ik\xi}. \tag{126}$$

It follows from (126) with regard for (125) that at $\xi = 0$ and $\xi = 2\pi$ the expression for matrix G (126) coincides with expression (118), which the matrix G has in the absence of artificial dissipator. It follows from (126) that the matrix G is a continuous function of variable ξ. Therefore, at $\xi \approx 0$ the behavior of quantity $\|G^n\|$ corresponding to scheme (123), (124) will be qualitatively the same as in the case of the scheme without the artificial dissipator. Thus, the introduction of dissipator (122) does not make the Lax–Wendroff scheme uniformly stable in the sense of Definition 1.

A comparison of the schemes considered in Examples 3 and 4 points to another possibility for the construction of uniformly stable difference schemes. We recall that the Runge–Kutta scheme (45) for the two-dimensional acoustics equations (42) was considered in Example 3. The numerical computations have shown that this scheme is not uniformly stable. However, after the symmetrization of the original acoustics equations, the corresponding Runge–Kutta scheme has become uniformly stable, because its amplification matrix has become a normal matrix.

Thus, the symmetrization of the original differential equations is one of the possible ways for constructing the uniformly stable difference schemes.

6. CONCLUSION

We have proposed two new criteria of the sufficiency of the von Neumann criterion for stability of difference schemes. These criteria are the extensions of the well-known and widely accepted condition for the uniform diagonalizability of the amplification matrix G. The merits of these criteria are their constructive character and the simplicity of their realization especially with the aid of symbolic computations on a computer.

In the second part of the present work, we propose new concepts, the uniform stability and the locally uniform stability. As shown above, these concepts are related to the oscillations arising in the numerical solution of many physical problems. The problem of the suppression of oscillations is of great importance especially in those problems, where there are instabilities and bifurcations of the solutions. The problem of laminar–turbulent transition is one of practical importance. As was noted in [30], besides the well-known distortion of the velocity field by the scheme viscosity, a new unpleasant phenomenon was revealed recently: the appearance of spurious oscillations in the flow. This effect is the principal one for the direct numerical modeling of laminar–turbulent transition, and it is unfortunately almost impossible to control it. The appearance of the spurious oscillations is especially dangerous for a rapid, “explosive” character of the turbulence onset, for example, in a plane channel. This leads to a significant error in the determination of the time and location of the turbulence onset and also distorts considerably the turbulent velocity field. It is also noted that distortions at small scales inevitably cause large-scale distortions. It is noted in [30] in conclusion that the problem of overcoming the effects of spurious oscillations is not solved and is a difficult problem.

Another class of problems, where the oscillations may distort the solution significantly, is gas dynamics problems with stagnation regions, which were already mentioned above. As was shown above, the stagnation zone may be a source of scheme oscillations. Because the gas flow is subsonic in the stagnation zone and in its neighborhood the scheme oscillations propagate along the characteristics to the overall computational region and can distort the solution significantly.

The spurious oscillations may cause certain difficulties also for the numerical solution of stationary problems by the pseudo-unsteady method.

The results obtained in the present work (Theorems 3 and 4) give constructive methods for the control of scheme oscillations. In contrast with the investigation of conventional stability, where it is sufficient to analyze the behavior of the maximum eigenvalue of the amplification matrix, it is necessary to analyze the overall spectrum for the investigation of uniform stability and locally uniform stability. This problem is nevertheless solvable by the available advanced means of computer algebra. In our opinion, another important result of the present work is that Theorems 3 and 4 give constructive criteria for the construction of efficient difference schemes, that is, such schemes whose oscillations either weakly affect the solution behavior (locally uniformly stable schemes) or are completely absent (uniformly stable schemes). In particular, it has been shown above that the symmetrization of the original differential equations is one of the possible ways for the construction of uniformly stable difference schemes.

Despite the fact that Theorems 3 and 4 were formulated for the difference schemes with constant coefficients, they can be applied also to difference schemes with variable coefficients in the approximation of frozen coefficients, as was demonstrated in Section 5 for the one-dimensional Euler equations. For the multidimensional fluid dynamics problems, the mathematics does not become too complicated. Only the execution of the corresponding analytic calculations becomes more complicated. As was mentioned repeatedly above, the computer algebra systems may prove to be very helpful here. The investigation of uniform stability of the advanced difference schemes for the multidimensional fluid dynamics problems is a separate task, which will be the subject of our further research.

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